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To teach modern algebra[1]

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We do not need to choose between traditional and modern mathematics; rather, they blend together naturally and organically.

OUTLINE AND SUMMARY

FROM THE CREATIVE EFFORTS of many teachers and scholars there is beginning to take shape a well-balanced program for a modern mathematics curriculum. It combines some of the more recent mathematical developments with the best of the traditional methods and topics. In this synthesis the postulational method emerges as a very valuable tool, which clarifies but does not replace the well-tested inductive methods. To illustrate the relation between these methods is one of the aims of this paper.

The development falls into three parts. Part I discusses five topics of algebra (two of which might be labelled "modern" and three "traditional"), with emphasis on their harmonious interplay. The inductive treatment of postulates in Part I prepares the way for the important distinction, introduced in Part II, between *pragmatic* and *abstract* postulational systems. It is suggested that good teaching, on the high school and junior college level, requires in general the use of the one and avoidance of the other. This leads to a recommendation, in Part III, regarding the college preparation of future teachers.

PART I

There is a compelling need, in the secondary school and the junior college, for materials which will give our students new perspectives, new horizons. I am not referring here to the need, mentioned so frequently in the newspapers of late, to bring the excitement of modern developments into the beginning algebra class—the need to show the beginner in algebra that he is learning to use the powerful tools of modern science. This is a real challenge to us as teachers; but I am referring here rather to another problem, which is not of a kind to "make the newspapers." It is this: that our students come to high school after some years of "cut-and-dried" arithmetic. Here I am using the farm boy's vernacular of "cut and dried" to remind us that there is a *routine* way to perform nearly every calculation taught in the arithmetic of grade school, so that original thinking is challenged primarily by the *applications*. When we take over the education of these youngsters many of them have responded to this state of affairs by developing a mental "set"—a conviction or expectation that very little thinking is needed in learning mathematics, except in its applications—that there will always be a set of rules and algorithms. It is important not to let the pupil carry this attitude into algebra. If we are to climb out of the rut of slavish memorization, surely this is the place to begin the climb. And it is here, I think, that a little taste of modern mathematics can work wonders. Let me list here a few ideas, some of them new, some of them seventeenth-century or older, which we can use to give insight into the nature of mathematics,

[1] Revised from a talk given on April 12, 1958, at the Annual Meeting of the National Council of Teachers of Mathematics, in Cleveland, Ohio. I wish to thank the referees for many helpful suggestions.

and to bring some warmth and enthusiasm into the learning process. It is hoped that the topics listed will show how well some of the modern mathematics blends with the essential parts of the traditional curriculum.

1. I believe we should emphasize the fact that there is not *just one* number system (the one made up of the integers and its extensions to rational numbers, real numbers, etc.), but that there are *many* number systems. Let us see how we can teach a new number system to boys and girls of age 10 to 15. (Of course, if we merely change our way of writing numbers, say by using the base twelve instead of ten, we are not getting a new system, but merely a new *notation* for our old familiar system. The new notation is important, and certainly interesting, but it is not what we want here.) Where can we find, then, a new number system? How do we begin the search?

Now there is a new number system which all of our students know, even though they may not know that they know it. I refer to "clock numbers." On a clock, $10+5=3$; $8+8=4$; and 2 nines are 6. (See Table 1.) I have found that students who had never responded, except with boredom, to the associative law for ordinary numbers, can get excited when we investigate whether $(7+8)+10$ is the same as $7+(8+10)$ in clock numbers. The fact that every additive linear equation ($8+w=3$; $7+w=11$; etc.) can be solved in this new system, without a need for "negative" numbers, gives them new insight. (If $8+w=3$, then $w=7$, etc.) The solution can be found in the table, or found by a manipulation. Further, they soon see that this solvability is related to the fact that, in each row of the addition table, each number occurs *once and only once*. Let us turn next to the multiplication table. Here, in the "3" row, we find that no matter what we multiply 3 by, we get either 0, 3, 6, or 9 as the answer. (We are using 0 for 12 here, as is customary and convenient.) Hence the equation $3w=5$ has *no* solution, while $3w=6$ has three solutions: $w=2$, 6, 10. (A kind of "com-

TABLE 1: Clock Numbers (*Note:* 0 replaces the 12 on a clock.)

Addition table

$a+b$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6												
7												
8												
9												
10												
11												

Multiplication table

$a\times b$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	1	6	11	4	9	2	7
6												
7												
8												
9												
10												
11												

pensation principle" works here; for there are as many extra solutions for some of the "lucky" equations as there are "unlucky" equations which lack solutions.) The failure of the cancellation law ($6\cdot 2=6\cdot 4$, but $2\neq 4$) in this system of clock numbers gives new meaning to an old cliché. And does $4(5+11)$ equal $4\cdot 5+4\cdot 11$ in clock numbers?

Students seem pleased to learn that there are new number systems, different from the one they have labored over. They like finite systems, like this one, in which they can get a bird's-eye view of the entire system, and can test the solvability of any equation whatever, if necessary, by just testing each of the 12 numbers, one at a time. Furthermore, they seem pleased to learn that one of the big jobs of mathematicians is that of setting up and studying *new* number systems. This viewpoint changes the mathematician from a kind of dogmatist, expounding an established, ancient doctrine, into a pioneer, who searches for systems of mathematics in response to human needs; and this makes him a more understandable and likable person in the eyes of the modern youngster. He becomes a creative explorer, working on man's intellectual frontiers, rather than an expounder of a system moldy with age.

Most important of all, perhaps, is the fact that clock numbers or other new number systems can shock students out of the "cut-and-dried" attitude, mentioned earlier, of waiting for someone to hand out the rule book which will take the place of thinking.

Clock numbers are called, technically, numbers *modulo* 12, or a number system of *modulus* 12. Our students can now, with a little help, construct other modular number systems (based on a clock with 5 hours, say, so that $5=0$). Some of them will discover for themselves that whenever the modulus is a prime number, the resulting system will be exceptionally convenient and easy to work with, for there will be a unique solution for each equation of the form $ax=b$, when $a\neq 0$; so that division (except by zero) is defined and unique. In fact, for any prime modulus the resulting number system is of the desirable type called a ***field.*** We can use here the method made famous in geometry by Professor Fawcett—that is, we can lead our students to discover that *some* of the main properties of these systems (solvability of $a+x=b$, cancellability, the uniqueness of the inverse of a number, etc.) can be proved logically from *others.* Thus we are led to single out certain properties as basic, and call them postulates. In this way we can introduce our students to the well-known set of postulates called the field postulates.

THE SYSTEM MODULO 5

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

I have taken time to discuss in some detail these modular number systems because I wanted to illustrate a *method,* and a *goal.* The method is that of introducing new mathematical ideas pragmatically, as growing out of experience. The goal is that of stimulating interest in the familiar, well-known mathematics which, it seems to me, will be for many decades the heart of our curricula. The student sees more clearly the importance of checking the answers to his equations, now that he has met systems in which linear solvability fails. He will have new respect for the theorem, "if a product of two numbers is zero, then one of the numbers must be zero," since this theorem fails for clock numbers. His appetite for proofs should be much sharpened, and even such an old story as the distributive law can command new respect.

Much remains to be said about the importance of teaching that there are *many* number systems, but this must be deferred to Part II, for it is time to move on to the next items.

2. We can point out to our pupils that whereas arithmetic provided many individual facts about individual numbers (for example, such facts as $3+5=8$, and how to divide 9640 by 88), it has left almost untouched the large field of investigating what *general truths* there are; truths which hold for all numbers. Our students know, of course, that any number minus itself equals zero, and that, in ordinary numbers, $x+y$ is always the same as $y+x$, etc.; and we can add many more truths to the list. Now we can show them the *generative power* of these general number truths. We can show them how, from $a(x+y)=ax+ay$, we can logically derive a law of squaring, $(x+y)^2=x^2+2xy+y^2$, and the law of fractions,

$$\frac{x}{z}+\frac{y}{z}=\frac{x+y}{z},$$

and many other results which are called, sometimes, the "laws of algebra." I omit the details, because they can be found in many places, such as in Professor B. E. Meserve's *Foundations of Algebra for High School Teachers*, and in some of his other writings; and in Professor B. W. Jones's *Elementary Concepts of Mathematics*. My point here is that we can easily convince the pupil that he is entering a new world of logic, a world of tremendous power, a world whose goal of discovering and using general truths (or *identities*, as we lamely call them[2]) is quite different from the aim of the arithmetic he has known. This new goal blends well with the study of the modular numbers discussed earlier.

3. But algebra is not concerned exclusively with general truths. It also busies itself with finding unknowns. Of this business of algebra I will only say that it seems to have captured the imagination of our students, and their parents and friends, until they see all algebra as nothing but a search for that elusive unknown. They even try to solve an identity, such as $3x+5x=8x$, for the "missing" x. One corrective for this is to emphasize the reading of such equations, occasionally, in English. Thus, $3x+5x=8x$ can be read as "3 times *any* number plus 5 times *that same* number always equals 8 times *that same* number." When our students see that such an equation is true for every number in the system, they will see that "solving it" could not give any additional knowledge.

If we emphasize, furthermore, that in large parts of algebra we are exploiting the remarkable ability of algebra to generate powerful identities almost effortlessly—identities such as $(x+3)(x-2)=x^2+x-6$, or

$$\frac{x^3-1}{x-1}=x^2+x+1$$

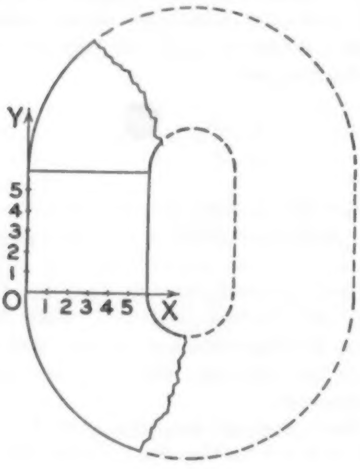
—and that in *these parts of algebra* we will not settle for anything less than general truths, then our students will not be disturbed and angry over the fact that they may be marked WRONG when they write $2x+3y=5xy$, and yet the next day be given this very equation and asked to graph it.

Here, as in the preceding item, I would like to point out that the relations and distinctions between different kinds of equations are seen in miniature, and more clearly, in a modular number system.

4. In the seventeenth century Descartes and others introduced the variable and the function into mathematics. (This is, of course, a historical oversimplification, but sufficient for our purpose.) I wish to stress the old-fashioned view that we can use these ideas in capturing the imagination of our students, and in showing them

[2] The term "identity" as applied to certain equalities does not distinguish "John = John" and "$5=3+2$" from *schema* such as "$x+x=2x$" which are true for all the values of x in the master set or number system being considered. (Of course, "$5=3+2$" can be replaced by "$5+0x=3+2+0x$," but this expedient seems repugnantly artificial to many beginners.)

how mathematics relates to modern science and engineering. After all, numbers are unchanging, yet mathematics applies to the changing world around us. How can this be? Let us illustrate the answer by drawing a part of a cross section of a jet engine (as you know, the word "jet" is a magic word with a large cross section of our clients) and consider the problem of an engineer who needs urgently to know the temperature distribution in this cross section. Before he can describe the temperature distribution, he needs *names* for the various points. "A," "B," "C," etc., are useful names for some purposes, but (*a*) there are too few of them to deal with an infinite number of points and (*b*) they do not have the structure properties that we need in a system of names. So we use a co-ordinate system, and name the points (2,1), (3,4), etc.



The next step will be to write the formula, or function, for temperature distribution. Here we must frankly admit that deriving this formula requires post-calculus work, but we can *invent* any formulas we wish for illustrations. If $T=700+5x+5y$ gives the temperature (in degrees Fahrenheit, say), the students can plot the temperature at each point of the "number map." They can join the points to find the isotherms. They can tackle the inverse problem: how to graph, say, the 715° isotherm; that is, how to graph $15=5x+5y$. Some students develop more enthusiasm for this kind of problem than for any other I have seen. The ideas of number distributions (temperature or pressure distributions and others), and of mathematics as a necessary tool for *describing* what goes on in nature, give them a new picture of the tie-up between mathematics and science.

I mention the mathematics of functions here chiefly as a background for a few brief remarks on the use of set theory in beginning algebra. It seems to me that the approach outlined above is a *warm* one. If, however, we begin by speaking of the substitution set of the variables, and the Cartesian product set in which the ordered pair (x, y) takes its values, and encumber each formula at the start with a domain and a range, then, I think, we are substituting a *cold* approach. Sets are unavoidable (not only in algebra, but in chemistry, history, and the game of bridge), but I believe the sophisticated language of set theory should be introduced very gradually indeed. Set theory is needed as soon as the student is ready for more precision; but first let us try to stimulate a genuine enthusiasm for the power of variables and formulas in describing nature, and in pure mathematics as well (as in analytic geometry). We can show how, for a falling body, the velocity and the time both change, but the formula $v=32t$ is an invariant feature of this changing situation. The formula selects the set of pairs (t, v) which apply to this falling body—and in saying so we use set theory, but we do not yet need, it seems to me, the formal symbols for union and intersection.

5. I have been asked to outline a method of introducing the theory of groups in a pragmatic, inductive manner. My choice for this purpose is the study of ways of shoving the living room furniture around; arrangements of the radio, sofa, and table,

for example. Here the student can draw pictures, or move three coins around, to verify the astounding fact that permutations of the furniture lead to a number system, one with "good" properties of solvability of linear equations, but one which is *non-commutative*. The *theory of groups* unifies large areas of mathematics, but if we tell our students this at the start they will expect something profound; and what is profound, they think, must be difficult. Instead, I use the following classroom material about a henpecked husband whose wife was fond of furniture moving—that is, fond of her husband's furniture moving. . . . His name is Adamson.

Imagine, then, an ordinary living room containing, among other things, a sofa, a radio, and a table. At the end of each hard day's work, Mr. Adamson comes home to find that his wife has decided to try a new arrangement of these three objects. She makes her "orders of the day" explicit by writing a note, which is always in the same form. A typical example is

$$\begin{bmatrix} RST \\ TRS \end{bmatrix}.$$

Mr. Adamson knows all too well what this means: "Put the table where the radio now is, and put the radio where the sofa now is, and put the sofa where the table now is." Mr. Adamson has received so many of these little messages that he has named them. He calls this one G, because it was first handed to him on green paper.

Mr. Adamson early learned that he is expected to replace the objects named in the top row by those in the bottom row, and not vice versa. The top symbols represent "before" and the bottom ones "after." He also noticed that there are several messages which look different but which contain the same instructions. For example, the message

$$\begin{bmatrix} SRT \\ RTS \end{bmatrix}$$

contains exactly the same instructions as the one he originally received on green paper, so he calls this one G also. In fact, when he tries to tabulate all possible messages, he finds only six basically different ones. One of these he looks at with special longing; it is the message

$$\begin{bmatrix} RST \\ RST \end{bmatrix},$$

which would require no work. Though he has never been granted this boon, he has included it for the sake of making his list complete, and has named it E (for "Easy"). The message

$$\begin{bmatrix} RST \\ STR \end{bmatrix}$$

he has named J, because he can still remember the hot June day when he would have liked to go fishing instead of moving furniture. The remaining ones he names alphabetically; he uses A for

$$\begin{bmatrix} RST \\ RTS \end{bmatrix}$$

(in which the radio is not moved), B for

$$\begin{bmatrix} RST \\ TSR \end{bmatrix},$$

and C for

$$\begin{bmatrix} RST \\ SRT \end{bmatrix}.$$

Now these names, G, J, and the rest, are only names, and you may be inclined to think they are not very important. A name is only a name; it moves no furniture. But by now you should be suspicious of this agnostic attitude. We have seen that a set of language agreements is an important part of a new system of thought, and that an apt naming system frequently leads to new insights. (If this were not so, mathematics would still be a branch of English!) So let us see what comes out of this system of names which Mr. Adamson gave to his wife's repeated orders.

As Mr. Adamson now recalls it, his interest in the implications of this little game first occurred to him when his wife's aunt became ill. One night Mr. Adamson came home and found the inevitable message (this time it was A) accompanied by a note which said "I am spending the night with poor Aunt Susan. Your dinner is in the refrigerator." Mr. Adamson was especially hot and tired that day, and, human nature being what it is, when the next evening came the furniture was still unmoved and another message (B) had arrived. Instead of actually moving the furniture twice, he decided to find out by means of pencil and paper what the final positions of the furniture should be after the two manipulations. He wrote an "R," an "S," and a "T" to indicate the original positions of radio, sofa, and table; and then wrote underneath them the letters which would occupy that position after manipulation A, repeating this process for manipulation B. He then inspected his list of the six possible manipulations to see which one would accomplish this all in one step, and discovered that G would do it.

$$\begin{array}{ccc} R & S & T \\ R & T & S \\ T & R & S \end{array}$$

He abbreviated this information by writing "$A+B=G$." He proceeded happily to perform the rearrangement G, quite pleased with himself for having "telescoped" two days' messages into one. From this time onward (believe it or not) he shirked his work and let the messages pile up until his wife was due to return home, at which time he added the accumulated messages. He found that, no matter what set of messages he had received, there was always one single manipulation which would produce the same furniture arrangement as the entire set of manipulations his wife had ordered.

After this discovery, he sought to avoid even the work of performing the additions by simply recording all his sums for future use, like an addition table. And finally (as often happens) this effort to avoid work was his undoing. His wife left him message B one Monday night, and left for Aunt Susan's; on Tuesday night she sent over message A; he recalled that he had already proved that $A+B$ is the same as G. He was just completing manipulation G on the furniture when his wife arrived and asked him, in icy tones, just what he thought he was doing. Flushed and unhappy, he worked out the separate messages on paper and found that message B followed by message A (which was the sequence his wife had ordered) gave a result different from that of message A followed by message B; in fact, $A+B=G$, while $B+A=J$! In this way he discovered that *addition of these messages is not commutative.*

TABLE 2

THE FURNITURE-MOVING GROUP

$$J=\begin{pmatrix} R & S & T \\ S & T & R \end{pmatrix} \quad G=\begin{pmatrix} R & S & T \\ T & R & S \end{pmatrix} \quad E=\begin{pmatrix} R & S & T \\ R & S & T \end{pmatrix}$$

$$A=\begin{pmatrix} R & S & T \\ R & T & S \end{pmatrix} \quad B=\begin{pmatrix} R & S & T \\ T & S & R \end{pmatrix} \quad C=\begin{pmatrix} R & S & T \\ S & R & T \end{pmatrix}$$

For example, J means "put the sofa where the radio is, the table where the sofa is, and the radio where the table is."

(Replace the item named in the *top* row by the one named in the *bottom* row, not vice versa.)

Addition of two elements, say $J+A$, means "find the result of performing the furniture arrangement J *followed by* the rearrangement A." We find this addition table:

		Y					
	$X+Y$	E	A	B	C	G	J
	E	E	A	B	C	G	J
	A	A	E	G	J	B	C
	B	B	J	E	G	C	A
X	C	C	G	J	E	A	B
	G	G	C	A	B	J	E
	J	J	B	C	A	E	G

Note these basic properties of this system:

1. Commutativity $(X+Y=Y+X)$ fails.
2. Associativity, $X+(Y+Z)=(X+Y)+Z$, holds.
3. Closure (every 2 arrangements have a sum) holds.
4. There is a neutral element, E. $(E+X=X.)$
5. Each element has an inverse, which "undoes" it, so that their sum is the "do-nothing" or neutral element, E. $(A+A=E, G+J=E$, etc.)
6. Note that $A+G=J+A$, but $G \neq J$. (Noncancellability, when different orders are involved.) However, whenever $A+X=A+Y$, then $X=Y$; as we see from the fifth statement above.[3]

In this exposition of a little group theory I have tried again to derive a system of mathematics from common experience. The student soon sees that such systems apply equally well to rearrangements of furniture, or molecules, or other objects or concepts; and he begins to appreciate the *generality* of mathematics. He begins to like the taste of things to which he previously may have been allergic; and an abstract mathematical system can never be quite so abstract again. We can go on to show him systems which are *isomorphic;* that is, systems whose apparent differences are due merely to our having chosen different names for the elements; and we can show how group theory underlies and unifies large areas of ordinary arithmetic. He will not be surprised, then, to learn that group theory has led to the most profound discoveries, from the theory of equations to quantum mechanics; and that much of mathematics has been reoriented around the concepts of finite and of continuous groups. The important concept of field, which has as an instance the rational number system, is easily defined in terms of groups and the distributive law.

[3] The above narrative about furniture-moving is adapted from a book now in press, *Foundations of Mathematics*, by C. H. Denbow and V. Goedicke, Harper & Brothers. The narrative leads up to a discussion of group postulates.

Part II

Many of the problems which confront teachers of mathematics today are by-products of the revolution in the philosophy of mathematics which has occurred in the last century or so. Questions which are perplexing today must have been rather easy to answer—once. For at that time there was only one number system, essentially, and it was believed to apply everywhere, to banking and to chemistry, to social science and to astonomy. Now we have myriads of number systems; and we are told by some eminent authorities that we must, in democratic fashion, regard them all as equals. We must consider that each is derived from a set of postulates, and that one set of postulates is as arbitrary as another. Somewhere between these two philosophical extremes we teachers must find a workable compromise. For not only is it impracticable to give the average ninth grader this latter point of view, but also it overlooks the intricate and important problem of the relation between an abstract formal system and the intuitive or pragmatic mathematical knowledge which the development of such a system presupposes—the knowledge that $2x$'s followed by $2x$'s is $4x$'s; that if one set of symbols pairs off with a second, and the second set with a third, then the first set will pair with the third; to mention only two examples. If we could not depend on basic assertions of this kind (and there are many of them), we would be forced to modify our formal development of mathematics—which is enough to show that postulational systems, in spite of their basic importance, are not self-sufficient as foundations. The analysis of this intuitive or pre-postulational mathematics, which underlies the use of formal systems, is of interest to many scholars, and should prove to be of great value to the working teacher.

We can avoid some of the communica-

tion breakdowns which center about the role of postulates in the teaching of mathematics if we distinguish carefully between two methods, both of which are frequently labelled "postulational."

1. I shall refer to an *abstract postulational system* as one which starts from undefined symbols and states formal postulates for the use of these symbols. All of its theorems and methods must follow logically from this basis.

2. By a *pragmatic postulational system* I mean a system which takes (an existing) body of knowledge, and traces out its logical relations, and finds that the body of knowledge can be deduced from a few basic principles or postulates. The third chapter of Freund's new book, *A Modern Introduction to Mathematics* (Prentice-Hall), illustrates what I mean. The discussion of field postulates and of group postulates provide further illustrations of this approach. The use of pragmatic postulational systems can make a great contribution to high school and junior college teaching. In my opinion, the extensive use of abstract postulational systems is unrealistic at this level, except in the most highly selective classes.

The difference between the two kinds of postulational systems can be illustrated in this way: From the abstract system point of view the distributive law and the other field postulates are arbitrarily chosen rules, to be applied to undefined symbols; while if we use ordinary numbers as an example of a pragmatic system, then these same field postulates become a storehouse or treasury for a huge portion of the accumulated mathematical wisdom of the human race.

Part III

How does all this relate to the college preparation of future teachers?

Point 1. The first college course that a future teacher takes is a crucial one. I believe it should teach modern topics, but in a pragmatic manner. It should avoid the danger of an overly mechanical, manipulative course on the one hand, and the danger of excessive abstraction on the other. It is not that I fear that the student cannot master the abstractions—the danger is that he will teach in the same vein in which he was taught. Professor Van Engen has stated a somewhat similar idea in reference to teacher-preparation courses, as opposed to mathematical courses for engineers: "There should be time to linger over ideas of special importance. There should be time for the teacher to make the thought-structure a part of himself."

Point 2. The teacher needs *thorough* preparation. The first, pragmatic course should be followed by many others in which he gains breadth and depth, in modern algebra, modern geometry, in calculus and analysis, in statistics, and in history and foundations.

These courses for prospective teachers should include the manipulations they will need to know, and also include strong and increasing emphasis on proofs, on the logical bands which keep mathematics from degenerating into a mere potpourri of rules. But these courses will not be adequate for the purpose unless they also bring a growing emphasis and appreciation of the main *ideas* of mathematics—the ideas of number systems, of identities, of limit, of a correspondence between algebra and geometry, of an invariant, and so forth. The prospective teacher should feel that he has a rich treasure house of worthwhile ideas, for then he will search enthusiastically for methods which will arouse his students' interest in these ideas. He will feel free to use careful proofs, or applications, or laboratory equipment, depending on circumstances. For every class he will be able to find the level of material which will arouse interest, and he will never miss a chance to improve their understanding of the logic of mathematics. This is a rather Utopian view of the teacher of tomorrow, to be sure, but can we settle for anything less?

*Normative decision models**

HOWARD RAIFFA, *Harvard University, Boston, Massachusetts.*

Decision making and game theory are two of a number of exciting new fields in mathematics. This paper gives a few simple illustrations of the newer developments and raises some thoughtful questions for the consideration of high-school teachers.

IT IS INDEED an honor for me to address this meeting of the National Council of Teachers of Mathematics—a group which has only recently gained its deserving recognition in the eyes of the nation. I guess the Sputniks have ignited the spark; the gears are now meshing and most everyone is jumping on the band wagon for "More and Better Mathematics." But some have warned that we had better have a driver in the forward compartment to steer the monster and to apply the brakes gently as we round sharp corners. I sense the excitement of an intellectual renascence. This distinguished body of pedagogues and scholars has a newly kindled mission, for now it has a mandate to partially shape the course of education in this "Age of Science."

The high school student faced with the problem: "Should I elect an optional mathematics program?" has a personal decision problem of vast complexity. If he chooses mathematics, in effect he must pay today an amount *certain* in sweat and frustration for a future delivery of an *uncertain* stream of rewards. It seems to me that for at least the more academically oriented youths the problem is becoming easier to solve. The "sure-thing principle" is taking over. The "sure-thing principle" states that if alternative A is preferred to alternative B for each potential state of uncertainty of the future world, then A is simply preferred to B. The alternative of taking a good deal of high school mathematics is preferred to the alternative of not taking this mathematics, not only for the future physicists, chemists, engineers, but also for the future economists, psychologists, sociologists, zoologists, anthropologists, business administrators, doctors, social workers, teachers—and, believe me, I could go on further! I think it goes without saying that future physical scientists should take a heavy dose of mathematics in high school. Some positive propaganda, however, is sorely needed to inform today's future natural, behavioral, and managerial scientists that they, too, are seriously hindering their future academic flexibility if they fail to learn more mathematics. How often I have heard the refrain from social and behavioral scientists, from public and business administrators and like groups, to wit: "If only I had taken more mathematics in high school, I would not have this sense of frustration of wanting and of needing to learn something about mathematics and of not having the opportunity, the time, or the stick-to-itiveness at this late date."

The late eminent psychiatrist Dr. Freda Fromm-Reichmann facetiously remarked to me that she felt in need of personal psychotherapy to help herself cope with

* Address delivered at the Thirty-Sixth Annual Meeting of the National Council of Teachers of Mathematics at Cleveland, Ohio.

her anxieties about her lack of sophistication in the uses of statistics in her own field (and let's not forget that statistics is a branch of applied mathematics!) Perhaps the Mental Health section of the Public Health Department should pass a strong resolution: "Resolved, that the egos and ids of our future generations need the nourishment of copious quantities of mathematical experiences." Now wouldn't that be a bit of normative decision making?

Now let us turn our attention from this horrendous sermon—which can liberally be interpreted as "normative influence making"—to the more academically traditional topic: "normative decision making," which incidentally is my topic today, in case you have forgotten.

As much as I am constitutionally opposed to classifying problems into neat, disjointed, and exhaustive compartments, I must admit this approach offers an easy way to get into a subject fast. However, I fear that the classification I will give of the area of decision making will not be neat but, rather, the boundaries will be hazy; the compartments will not be disjointed, but the overlaps will be only too obvious; and my breakdown will be far from exhaustive—indeed, it would be easy for me to play devil's advocate against myself by asserting that I am singling out a very small part of the area—but I hope not an insignificant part!

Let's first chop off a tremendous and important field which we will not further subdivide. This is the area of *descriptive decision making*—an area whose literature is becoming increasingly mathematical—an area which deals with what decisions people do in fact make, how and why they make them, how they learn, how they rationalize to themselves that their chosen action was quite clever after all, etc. Instead, I will confine my attention to *normative* or *conditionally normative* decision making—that is, how people "should behave" or "would like to behave" in order to achieve certain prescribed ends. Or, better yet, how *I* would like to behave. Not that I hold myself as being more "rational" than the next, but at least I can talk for myself. After all, "rationality" is an awfully subjective affair.

In many practical problems of the business world, we get a nice blending of descriptive and normative decision making. The entrepreneur wishes to act "rationally" and "consistently" in a competitive business situation, and to do so he must painstakingly study how people (consumers and competitors alike) do in fact make choices, how they can actually be influenced, etc. How *should* the entrepreneur behave in order to exploit how people actually *do* behave?

Let's first consider the case of the *single individual*. We assume the individual faces the following decision problem: He must choose a specific act or strategy from a domain of feasible acts or strategies in order to "optimize his satisfaction."

Conceptually, the simplest case arises when each act x can be evaluated by an index $y(x)$, say, which reflects the relative desirability of x. One then chooses $x^{(0)}$, in the available domain, to maximize or minimize the function y. When x is a real variable or a set of real variables, and y is differentiable and the optimum x^0 value lies in the interior of the feasible domain, then the usual calculus techniques apply. Problems of this variety are quite common and the calculus techniques have been extensively applied.

Another class of problems that have been with us a long time but have only recently been formally treated goes under the title of *linear programming*. In this case, a strategy x formally becomes an n-tuple $(x_1, x_2, \cdots, x_n)$, i.e.: the act stipulates a set of n levels. The components of the n-tuple are restricted by a set of linear inequalities and $y(x)$ is a linear function. The feasible domain now becomes an n-dimensional polyhedron and the optimal x^0 is at a vertex of this polyhedron. The calculus has little to say about these problems and the mathema-

tician's aspirations now shift from a desire to get closed analytical solutions, to a desire to get algorithms, which can be programmed on digital computers. The importance of this shift of emphasis should be duly recognized, for its operational implications may change—aye, are changing—the nature of applied mathematics as we know it today. Just a few of the many applications of linear programming that have been made in the last decade are: applications to production and transportation scheduling problems, personnel assignment problems, mixing and blending problems, optional allocation of resources, etc. I hope that elementary aspects of some of this material will eventually sift down to the high school level, for more work with linear inequalities would be desirable and here is some opportunity to do maxima problems without having to make use of the calculus.

Now let us generalize the class of problems. Our hero now has to choose a strategy x from an appropriate domain, but now we assume the consequence of x is a bit more complicated. No longer do we assume that x leads to an outcome certain, but now we assume it can lead to one of several possible outcomes—which one, in particular, does occur depends on several other factors (external to our decision) which we lump together and call "chance." For example, in an inventory problem, the strategy might involve the specification of certain minimum and maximum stock levels for a given commodity and the consequence of any strategy might depend on the unknown state of future demands for the commodity. Or, in a medical diagnostics problem, the strategy might consist of following a specific course of experimentation with a decision rule stating explicitly what terminal diagnosis to make for each potential experimental eventuality. The appraisal, of course, of any such strategy depends—in this case—on the unknown *state* of the patient's affliction.

In order to grapple with this new class of problems, let us make the simplifying—but not really binding—assumption that a strategy, such as x, leads to one of a finite set of outcomes, viz.:

$$\text{Strategy } x \xrightarrow{\text{(leads to)}}$$

$$\left\{ \begin{bmatrix} \text{Outcome } X_1 \\ \text{if} \\ \text{state } S_1 \text{ prevails} \end{bmatrix}, \begin{bmatrix} \text{Outcome } X_2 \\ \text{if} \\ \text{state } S_2 \text{ prevails} \end{bmatrix}, \cdots, \begin{bmatrix} \text{Outcome } X_r \\ \text{if} \\ \text{state } S_r \text{ prevails} \end{bmatrix} \right\}.$$

We assume one and only one state will occur; i.e., $S_1, \cdots, S_r$ is a mutually exclusive and exhaustive partition of certainty. An outcome such as X_1 might be a complicated stimulus encompassing all the economic, psychological, and sociological factors involved. One point of view holds that one should not make a detailed analysis of situations of this kind; that the decision maker should merely choose amongst strategies as total entities; that what is needed is a gut reaction to the Gestalt associated with one strategy versus the Gestalt associated with any other. I do not share this viewpoint, but I do hold that a naïve analysis in situations of this kind can do more harm than good.

In the past decade a great many researchers in such divergent fields as mathematics, psychology, economics, and philosophy have specialized in such problems of risky choice. The modern literature in this field, which is now quite sizeable, can be said to have started with some of the pioneering work of the late eminent mathematician, John von Neumann, who resurrected the then-defunct notion of classical utility, so ably misused by the economists of old, and put a variant of this notion on a precise and firm mathematical

base. Although von Neumann's theory of utility was tailor-made by him to fit in with his then-blossoming theory of games, the theory of utility can stand by itself and it promises to have even more widespread application than some of the more famous parts of game theory proper.

Let's take a very brief look at von Neumann's theory of utility. First of all, it makes one big assumption, to wit: although a strategy *cannot* a priori be said to lead to a specific outcome certain, it can be said to lead to one of a known set of possible outcomes, and the *probabilities* of these outcomes are known. I repeat . . . the *probabilities* of the outcomes are known. We have, for example,

$$\text{Strategy } x \xrightarrow{\text{(leads to)}}$$

$$\underbrace{\left\{\begin{bmatrix}\text{Outcome } X_1\\ \text{with}\\ \text{probability } p_1\end{bmatrix}, \cdots, \begin{bmatrix}\text{Outcome } X_r\\ \text{with}\\ \text{probability } p_r\end{bmatrix}\right\}}_{\text{Lottery associated with strategy } x.}$$

Thus to each strategy there is associated a lottery which stipulates a probability measure over a set of outcomes—or *prizes*, if you will. Conceptually the problem reduces to analyzing the decision maker's preferences for diverse lotteries. The theory now says that if a subject's preference pattern (over a sufficiently rich class of alternative lotteries) satisfies some consistency requirements, then real numbers can be associated with the outcomes in such a manner that the expected values of these numbers become a suitable index to maximize. (For example, if $r=3$ and X_1, X_2, X_3 get assigned values 1, .8, .3, respectively, and if $p_1=.4$, $p_2=.5$, $p_3=.1$, then, symbolically,

$$\text{Strategy } x \xrightarrow{\text{(leads to)}}$$

$$\left\{\begin{bmatrix}X_1\\ p_1=.4\end{bmatrix}, \begin{bmatrix}X_2\\ p_2=.5\end{bmatrix}, \begin{bmatrix}X_3\\ p_3=.1\end{bmatrix}\right\},$$

which has an associated (utility) index,

$$y(x)=1(.4)+.8(.5)+(.3)(.1)=.83.)$$

The numbers assigned to the prizes are called *utilities* and are unique up to positive linear transformations—in the same way that temperatures, for example, are meaningful up to positive linear transformations of scale. The theory essentially tells us how to assign these numbers, which in a pithy manner tautologically mirror the decision maker's preferences for risky lotteries. Now let me play a question-and-answer game with myself.

Question 1: If people were asked to make preferences for complex lotteries would they be consistent in the sense of von Neumann?

Answer 1: No!

Question 2: Would I?

Answer 2: No!

Question 3: Would I like to be?

Answer 3: Yes!

Question 4: So where does that leave us? What's the punch line?

Answer 4: Well, taken as a descriptive theory the model doesn't fit reality. However, the fit might be close enough, for some descriptive purposes. But I wish to stress the *normative* implications of the model. Instead of exhibiting my stupid inconsistencies by boldly stating my preferences for all sorts of complex lotteries, I will only venture to commit myself on how I feel about relatively simple lotteries—some of these might be quite hypothetical and not at all available in the realistic decision problem. After committing myself to choices among the simple cases I now *force* myself to be consistent by mathematically computing what my reactions have to be when faced with choices among complex lotteries. In essence, what I'm doing is: decomposing complex lotteries into a series of simple ones, registering my subjective preferences for the simple alternatives, and pasting these together again with a type of cohesive glue which incorporates the con-

sistency requirements which I deemed desirable in the first place. I am bringing a mathematical tool to bear on an intrinsically psychological, subjective measurement problem by psychoanalyzing myself, so to speak.

A simple illustration might aid in setting forth the essentials of the utility concept. Mr. A is given two lotteries with monetary prizes to choose between. The choice is a one-shot affair not to be repeated. The following display will be discussed below:

LOTTERY 1			EQUIVALENT LOTTERY		
PROBABILITY	MONETARY		−\$100	\$100	UTILITY
.3	−\$100	~	(1.0	0.0)	$a+.0b$
.2	0	~	(.3	.7)	$a+.7b$
.2	+ 50	~	(.1	.9)	$a+.9b$
.3	+ 100	~	(0.0	1.0)	$a+1.0b$
	\$10		(.38	.62)	$a+.62b$

1*a*) Expected *monetary* value of lottery 1 = .3(−100) + .2(0) + .2(50) + .3(100 = 10.

1*b*) Lottery 1 is equivalent to: $\begin{pmatrix} -100 & 100 \\ .38 & .62 \end{pmatrix}$

—i.e., to the lottery which results in prizes −\$100 and +\$100 with probabilities .38 and .62, respectively.

1*c*) Expected *utility* value of lottery $1=.3(a+.06)+.2(a+.7b)+.2(a+.9b)+.3(a+1.0b)$ $=a+.62b$.

LOTTERY 2			EQUIVALENT LOTTERY		
PROBABILITY	MONETARY		−\$100	\$100	UTILITY
0	−\$100	~	(1.0	.0)	$a+.0b$
.6	− 50	~	(.5	.5)	$a+.5b$
.1	+ 50	~	(.1	.9)	$a+.9b$
.3	+ 100	~	(.0	1.0)	$a+1.0b$
	+\$5		(.31	.69)	$a+.69b$

2*a*) Expected monetary value of lottery 2 = .0(−100) + .6(−50) + .1(50) + .3(100) = 5.

2*b*) Lottery 2 is equivalent to: $\begin{pmatrix} -100 & 100 \\ .31 & .69 \end{pmatrix}$

—i.e., to the lottery which results in prizes −\$100 and \$100 with probabilities .31 and .69, respectively.

2*c*) Expected *utility* value of lottery $2=.0(a+.0b)+.6(a+.5b)+.1(a+.9b)+.3(a+1.0b)$ $=a+.69b$.

Lottery 1 has the larger expected monetary value, \$10 versus \$5. Does this mean Mr. A should prefer lottery 1? Not at all! Lottery 1 might entail a loss of \$100, whereas with Lottery 2, A is sticking his neck out for a maximum possible loss of \$50. Suppose A, instead of reacting to the complicated paired comparison between the two lotteries, pries into his soul to get simpler "answers" to more basic questions. Suppose upon reflection he states:

1. He would be indifferent between getting nothing (\$0) and participating in a lottery with prizes $-\$100$ and $+\$100$ with probabilities .3 and .7 respectively. Symbolically,

$$\$0 \sim \begin{pmatrix} -\$100 & +\$100 \\ .3 & .7 \end{pmatrix}.$$

(See the correlate in the display.)

2. He would be indifferent between getting \$50 certain and participating in a lottery with prizes $-\$100$ and $+\$100$ with probabilities .1 and .9 respectively. Symbolically,

$$\$50 \sim \begin{pmatrix} -\$100 & +\$100 \\ .1 & .9 \end{pmatrix}.$$

3. Using a similar notation,

$$-\$50 \sim \begin{pmatrix} -\$100 & +\$100 \\ .5 & .5 \end{pmatrix}.$$

Substituting the equivalences stated in (1), (2), and (3) above in Lotteries 1 and 2, statements (1*b*) and (2*b*) follow. The .62, for example, comes from:

$$.62 = .3(.0) + .2(.7) + .2(.9) + .3(1.0).$$

Strategically, the problem boils down to the display below:

Since obviously

$$\begin{pmatrix} -100 & +100 \\ .38 & .62 \end{pmatrix}$$

is less preferred than

$$\begin{pmatrix} -100 & +100 \\ .31 & .69 \end{pmatrix}$$

—the latter having a higher probability on the best prize and a lower probability on the worst prize, and furthermore, the prizes being common to these two lotteries—Mr. A concludes that Lottery 1 is less preferred than Lottery 2.

Instead of going through the reduction to the basic equivalent lotteries involving the $-\$100$ and $+\$100$ prizes, Mr. A could equivalently give a utility score to the basic prizes, such as a to $-\$100$ (where a is arbitrary), $a+b$ to $+\$100$ (where b is an arbitrary positive number), $a+.7b$ to \$0 (see display), etc., and then work with the expected utility values of the lotteries. From (1*c*) and (2*c*) one sees that lottery 2 is more preferred—provided (and here is the crux!) that Mr. A wants to act consistently with his basic preferences recorded in (1), (2), and (3) and deems the manipulations legitimate.

The above theory of risky choice hinges on the basic assumption that probability assignments can be given to the respective outcomes of any lottery. In practice, how are these assignments made? Here we enter into a philosophical fracas. There are some probability assignments which belong in the public domain, so to speak. Here, no matter what school of thought one belongs to, specific numerical prob-

$$\begin{array}{ccc}
\text{Lottery 1} & \longleftarrow \text{(versus)} \longrightarrow & \text{Lottery 2} \\
\uparrow & & \uparrow \\
\text{indifferent to} & & \text{indifferent to} \\
\downarrow & & \downarrow \\
\begin{pmatrix} -100 & +100 \\ .38 & .62 \end{pmatrix} & \rightarrow\text{(is less preferred than)}\rightarrow & \begin{pmatrix} -100 & +100 \\ .31 & .69 \end{pmatrix}.
\end{array}$$

ability assignments are deemed appropriate and specific values are not subject to serious debate. But when one steps out of these narrow confines—essentially where past empirical frequencies and symmetries are not operative—everything breaks loose!

We are now in the domain of modern statistics—which deals with (normative) decision-making in the face of the unknown. The most common school of statistical thought, an objectivist school, typically partitions the domain of the unknown of an inference problem into two parts: one gets the usual probabilistic treatment; the other is shielded from the probability calculus. Another school, a subjectivist school, holds that no aspect of the unknown should be immune from a probabilistic analysis. For example, suppose it is suspected that a patient might have some form of tuberculosis. An experimental program is devised for this patient, and both schools may assign probabilities to the possible outcomes of the experimentation, *conditional* upon the patient being T.B. and conditional upon the patient being non-T.B. The subjectivists want to go further. They assert that a probability assignment should be given to the proposition, "This particular patient is T.B.," *ex ante* looking at the experimental results. This assignment is to be gleaned from the betting odds the decision maker would subjectively give to this proposition if confronted with a side gamble. The odds should be based on all the vague and subjective information that can be brought to bear on this particular case. These a priori probability assignments are then revised when experimental evidence becomes available.

The subjectivist viewpoint gets bolstered by its proponents' attempts to prove that if the decision maker wants to be consistent according to such-and-such so-called reasonable desiderata, he essentially *must* behave in this way. Of course, a good many bright fellows don't think that the so-called reasonable consistency desiderata are *really* reasonable. Personally, I'm sold, and I wish the subjectivist boys bon voyage in their arduous propaganda trek to come. They have some mighty convincing to do.

Another simple illustration might help. Mr. B is presented with the lottery L, described at the foot of this page.

Mr. B does not know very much about the stock market, but nevertheless he *must* give a monetary certainty equivalent for this uncertain venture. The subjectivists would have Mr. B proceed as follows: They would have him compare L with a more objective lottery,

$$L' = \begin{pmatrix} -\$100 & -\$20 & +\$500 \\ p_1 & p_2 & p_3 \end{pmatrix},$$

say, where $p_1+p_2+p_3=1$, $p_i \geq 0$, $i=1, 2, 3$. For example, if $p_1=.3$, $p_2=.5$, and $p_3=.2$, then L' could represent the following venture: Ten balls labelled 1 to 10 are placed in an urn. A single random drawing is made. Mr. B pays \$100 if ball 1, 2, or 3 is drawn; he pays \$20 if ball 4 to 8 is drawn; he receives \$500 if ball 9 or 10 is drawn. The lottery L' is manipulated by changing the p_1, p_2, and p_3 until Mr. B is convinced he is indifferent between receiving L and L'. Now, instead of trying to get a monetary certainty equivalent for L, Mr. B tries to do this for the indifferent L'. In such a manner Mr. B effectively

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LOTTERY L

Prizes:	−\$100	−\$20	+\$500
Events:	if tomorrow's Dow-Jones average goes down at least 4.0 points.	if tomorrow's Dow-Jones average stays within −3.9 to +1.9 points of today's value.	if tomorrow's Dow-Jones average goes up at least 2.0 points.

processes his vague subjective feelings about which way he thinks the market will go. Now the reasoning about L' can be decomposed further in a manner described above when we discussed risky choices with objective probabilities. If L is indifferent to L'—with $p_1 = .2$, $p_2 = .5$, $p_3 = .3$, say—then we can elliptically say that, *as far as decision making goes*, Mr. B's present subjective feelings about the likelihoods of the three critical events are captured by the weights or "subjective probabilities" .2, .5, and .3.

Let's move over to the next alley. In this alley there is more than one bright guy and each is looking after himself. Each fellow is trying to maximize his own satisfaction. However, we do allow the possibility that one's own satisfaction index might have an altruistic component. We now enter the realm of interacting decisions. In this realm there is a fantastic array of intriguing mathematical problems—which can be loosely herded together under the title Theory of Games.

The modern mathematical approach to interest conflict—game theory—is generally attributed to von Neumann in his papers of 1928 and 1937. This work lay dormant until the publication in 1944 of von Neumann and Morgenstein's monumental work: *Theory of Games and Economic Behavior*. Initially there was a naïve band-wagon feeling that game theory solved innumerable problems of sociology, political science, and economics —or that, at the least, it made their solution a practical matter of a few years' work. Nonsense! I would say that game theory has, however, focused attention on a class of conflict problems which is relevant to several major disciplines. The mathematical theory points its finger at a good many burning questions which are extremely relevant to the world we live in. To give answers, the theory has to divide out some difficulties, make idealized assumptions, etc., so that it would be naïve to apply the model intact to many real world-conflict situations. Nevertheless, the issues and concepts raised by the theory have their correlates in the real world and experience gained in the abstract world sharpens one's comprehension about real-life problems of the first magnitude.

Roughly, the abstraction is this: Each player in the conflict situation is required to choose one strategy from a well-defined set of possible choices. Given the strategies of each of the players, there is a resulting outcome which is appraised by each of the players according to his own peculiar tastes and preferences. Each player's preference pattern is summarized by his utility index which he wishes to maximize. The problem for each player is: What strategy should he choose in order that his partial influence over the outcome benefits him most? He, of course, must assume that each of the other players is similarly motivated. Each has to think about what the other fellow is thinking and about what the other fellow is thinking about *his* thinking, etc.

The social mores in most parlor games almost always specify that there shall be no collusion among the players of the game. Not so in the economic, political, and diplomatic worlds. Game theory lives up to these realistic necessities of life. A great deal of attention is paid to the forces which give rise to collusive actions and to the formulation of coalitions. It painstakingly worries about the external actions of coalitions and what their internal contractual arrangements must be for the group to remain viable and stable.

Often it is felt that conflicts of interest should not be allowed to resolve themselves in—shall we say—the open market of threats and counterthreats, but that there should exist social devices to take into account the preferences and strategic potentialities of each of the players and to arrive at a "fair" resolution of the conflict. Abstract conflict situations arising in the theory of games afford a host of potential test cases for laboratory experimentation with devices of social arbitration and conciliation. Such experimental work

has been done and more is needed.

Well, here we have it: Game theory is a model for situations of conflict in which such modes of resolution as collusion, coalition formation, conciliation, and social arbitration are not overlooked. I like the positive sound of the title of a book that R. B. Braithwaite, English philosopher, has recently published. The title is *Theory of Games—Tool for the Moral Philosopher.*

The area represented by the theory of games, by statistical decision theory and related decision-making models, is bubbling over with activity. So far the real innovators have primarily been pure mathematicians turned applied—but not necessarily in the direction of the physical sciences. It has not been a question of "Here is a new tool like topology—let's look for applications," but rather, "Here is a class of problems not previously held to be amenable to any sort of mathematical analysis—let's try to develop new tools to handle these new problems." What sort of mathematics does one need to read this literature? That's hard to say. Technically speaking, a great deal of the literature does not use mathematics beyond that to be found in high-school curriculum. But then again, students who have completed calculus and differential equations and have a high degree of manipulative skill find that they cannot cope with the frequent long trains of logical arguments. They have the manipulative skill but lack *mathematical maturity and sophistication.* Can this be taught? This, to my prejudiced viewpoint, is the great challenge to the modern mathematical educator.

What's new?

BOOKS

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Mathematics and Your Career, U.S. Department of Labor, Bureau of Labor Statistics, Washington 25, D.C. 10 pp., free.

*How I came across the extraction of "Nth" roots**

DAVID SHAPIRO, *Merrill Junior High School, Oshkosh, Wisconsin.*

An example of how a seventh-grade student's sense of pattern led him to generalize the method for finding the "Nth" root of a number.

I AM David Shapiro. I am in the seventh grade and I have a mathematics teacher who lets me stay after school and discuss mathematical problems. His name is Mr. Fuller.

At the beginning of this school year I got interested in square roots, and I asked my father to tell me how to extract them. He told me how to do it.

I told Mr. Fuller, one night after school, that I knew how to extract the square roots of numbers. We looked up some numbers in the square tables, and square-rooted them to see if the tables were right. They were.

Knowing how to extract square roots, I became interested in cube roots, and I asked Mr. Fuller if he knew how to extract them. He told me that it started out like square roots, except that all the twos were threes and all the squares were cubes. For instance: first in the square roots, you mark off every second digit from the decimal point; and in cube roots you mark off every third digit from the decimal point. Then he told me that in square roots, you take the nearest square root of the first set of numbers and put this in the answer. In cube roots you do the same thing with the cube root instead of the square root. Then in both cases you square or cube it, as the case may be, and subtract this from the first set of numbers.

* David Shapiro's article was printed in the October, 1958 issue of *The Wisconsin Teacher of Mathematics* and is reprinted with the kind permission of its editor, Carroll Flanagan, Whitewater State College, Whitewater, Wisconsin.

Then in both cases, you bring down the next two or three numbers, whichever it is. Then in square roots you double the answer so far and multiply it by ten; this you divide into the difference and put this digit as the next number in the answer. Then in square roots you multiply the divisor in the above division by the quotient, and to this product you add the quotient squared. This sum you subtract from the difference. Then you bring down the next two numbers, double the answer so far and multiply by ten, and so on as before. In cube roots, he said that he did not think that you tripled the answer so far and multiplied it by ten and divided this into the difference because in a number very close to a perfect cube, like 999,999, it often went over a thousand times, which is by far not a digit. Then we changed the subject and talked about something else and cube roots were forgotten for a while.

One day after Christmas vacation, Mark Clausen (a classmate also interested in mathematics) came in after school and said that he got a book for Christmas that told how to extract cube roots. We looked it over and tried it out. It said that after you had done everything the same as in square roots with all the twos, threes, and all the squares, cubes up to the point where you supposedly tripled the number, instead you add 300 times the answer so far and 30 times the answer so far squared, mentally, and divide it into the difference. The quotient you would add in with 300 times the answer so far and 30 times the

answer so far squared, and also it would become the second digit in the answer. Then you would multiply the above sum by the above quotient and subtract this from the difference, bring down the next three numbers, and repeat the process.

We tried this formula out, but it only seemed to work with the ones in the book, for when we tried this out with an almost perfect square, such as 999,999, it would go as many as 52 times, which was still not a digit. It was not far off, however, as I was soon to find out.

Mr. Fuller, a few days later, looked up the word "arithmetic" in the *Encyclopedia Britannica* and found that it told about the extraction of cube roots. It said that you start out as in square roots (as previously described) until you get to the point where you supposedly tripled the answer so far and divided it into the difference. It said that instead you would use this formula:

$$3T^2 \text{ plus } 3T,$$

where T equals the answer so far times 10. Then you would divide this into the difference and this digit becomes the next digit in the answer and also becomes U in the following formula:

$$3T^2U \text{ plus } 3TU^2 \text{ plus } U^3.$$

This you would subtract from the difference and bring down the next three numbers and repeat the process.

We tried this out on an almost perfect cube, such as 999,999, and it went a little over 10 times, but we carried it out as 9 and the answer was "O.K." when we added 9^3 to the number.

Now we had a formula for cube roots and I wanted to see what the mistake was in Mark's book, so I said, "Since in the formula

$$3T^2U \text{ plus } 3TU^2 \text{ plus } U^3$$

T equals 10 times the answer so far, let it equal $10A$, where A equals the answer so far." It looked like this:

$$3(10A)^2U \text{ plus } 3(10A)U^2 \text{ plus } U^3,$$

which reduces to:

$$300A^2U \text{ plus } 30AU^2 \text{ plus } U^3.$$

Then Mark's book said this:

$$U(300A \text{ plus } 30A^2 \text{ plus } U),$$

which is the same as:

$$300AU \text{ plus } 30A^2U \text{ plus } U^2.$$

Comparing the two formulas,

$$300A^2U \text{ plus } 30AU^2 \text{ plus } U^3$$

and

$$300AU \text{ plus } 30A^2U \text{ plus } U^2,$$

you can see that either the author made a mistake in calculating, or the publisher made a mistake in copying.

Then I figured a formula similar to the ones in the encyclopedia except for square roots. They were:

$$2T \text{ and } 2TU \text{ plus } U^2$$

where $2T$ is the answer so far doubled, times 10, and $2TU$ plus U^2 is double the answer so far, times ten, plus the new digit (which would be $2T$ plus U) times the new digit (which would be $2TU$ plus U^2).

Then I took the formulas for square roots and put them above the ones for the cube roots to see if there was any definite progression. It looked like this:

$$2T \cdots 2TU \text{ plus } U^2$$
$$3TU \text{ plus } 3T \cdots 3T^2U \text{ plus } 3TU^2 \text{ plus } U^3.$$

I then guessed that the ones for the fourth and fifth roots might look like this:

$$4T^3+4T^2+4T \cdots 4T^3U +4T^2U^2+4TU^3+U^4$$

$$5T^4+5T^3+5T^2+5T \cdots 5T^4U +5T^3U^2+5T^2U^3+5TU^4+U^5$$

and that the formula for the "Nth" roots might look like this:

$$NT^{N-1}+NT^{N-2}+NT^{N-3} \cdots NT^2+NT$$

and

$$NT^{N-1}U+NT^{N-2}U^2 \cdots +NT^2U^{N-2} +NTU^{N-1}+U^N.$$

I tried this formula out on even fourth powers, but for some reason I would come out with the remainder in even hundreds. For instance, when I took the fourth root of 11^4 I came out with a remainder of 200, and when I took the fourth root of 12^4 I had a remainder of 1800. I wondered what had happened, and I checked the work and the formula, but both seemed all right.

The next night after school, I told Mr. Fuller what had happened, and he suggested trying it on an algebraic expression. He said that the formulas should work if T equals the answer so far and U the new term. Thus we took the fourth root of:

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4,$$

which should come out as $a+b$, but it always came out as $a+b$ with a remainder of $2a^2b^2$. We added this into the formula for fourth roots (changing a to T and b to U), and we got this:

$$4T^3U+6T^2U^2+4TU^3+U^4,$$

and then I noticed that the coefficients of the terms lined up made this:

4641

which is practically 11^4. 11^4 is 14,641, as I remembered it from the time I used the other formula to extract the fourth root of 11^4. I also noticed that the coefficients of the other formulas lined up made this:

For cubes 331, and for squares 21.

When you put a "1" in front of these, you have 1331 and 121, which are 11^3 and 11^2. Then, by further experimentation, I figured the formula for fifth roots:

$$5T^4U+10T^3U^2+10T^2U^3+5TU^4+U^5.$$

But 11^5 equals 161051. I wondered what was wrong for a few minutes, until I noticed that "10" was not a digit and that "one hundred and fifty-ten thousand, ten hundred and fifty one" is the same as 161,051. Therefore, when figuring out 11 to any power to find the coefficients for an "Nth" root formula you should not do any carrying, and it doesn't matter if there is a 10 or higher number as a digit.

However, it is rather awkward to say, "Raise 11 to the Nth power without carrying," so we started to look for some definite progression in powers of 11 besides one being 11 times the one before. When we started out with 11 and put 121 under it and 1331 under that and 14641 under that and tried to figure what the next one would be, we noticed that each number is the sum of the numbers directly above it. It looked like this:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

We could also put a "1" above the eleven if we wanted to, but it would make the following formula false.

To extract the Nth root of a number, first mark off every Nth digit from the decimal point, then take the nearest Nth root of the first set of numbers and put this in the answer, then raise this number to the Nth power and subtract it from the first set of numbers, and bring down the next set of numbers. Then use the following formula (where T is the answer so far, times ten, and the X's will be explained after the formula):

$$X_1T^N+X_2T^{N-1}+X_3T^{N-2}\cdots$$
$$\cdots X_{N-2}T^2+X_{N-1}T.$$

If you lined up all the X's and annexed an X_0 at the beginning and an X_N at the end you could substitute the numbers in the Nth row in the table below for the X's.

1st row	1 1	
2nd	1 2 1	A B
3rd	1 3 3 1	C
4th	1 4 6 4 1	
5th	1 5 10 10 5 1	A plus B equals C
6th	1 6 15 20 15 6 1	$C-B$ equals A
		$C-A$ equals B

The small table at the right could be placed anywhere on the big table and should be used to extend the large table if it is not big enough.

After the above formula is figured out, divide it into the difference and this digit becomes the next digit in the answer and also U in the following equation:

$$X_1T^{N-1}U+X_2T^{N-2}U^2\cdots$$
$$\cdots X_{N-1}TU^{N-1}+U^N.$$

The meaning of the X's in the above equation was explained earlier.

However, although this is a better formula than the previous one, there is still one that doesn't need the use of any tables or awkward explaining. It is a little bit more complicated, however. Here are the two formulas:

$$NT^{N-1}+\frac{N(N-1)}{2!}T^{N-2}$$
$$+\frac{N(N-1)(N-2)}{3!}T^{N-3}$$
$$+\frac{N(N-1)(N-2)(N-3)}{4!}T^{N-4}+\cdots$$
$$+NT,$$

and:

$$NT^{N-1}U+\frac{N(N-1)}{2!}T^{N-2}U^2$$
$$+\frac{N(N-1)(N-2)}{3!}T^{N-3}U^3$$
$$+\frac{N(N-1)(N-2)(N-3)}{4!}T^{N-4}U^4+\cdots$$
$$+NTU^{N-1}+U^N.$$

In the above formulas, a number with an "!" after it means that all positive whole numbers below it are to be multiplied together with itself. For instance;

3! equals 1 times 2 times 3
4! equals 1 times 2 times 3 times 4
N! equals 1 times 2 times 3 times 4 times $\cdots$ $N-1$ times N.

The above formulas for Nth roots may be used on algebraic expressions with the following differences:

T equals everything so far.
U equals the new term.

Also, in the beginning you don't mark off every Nth term, and when you subtract you bring down all the rest of the expression, and when you divide the first formula into the difference you only divide the first term of the divisor into the first term of the difference.

Also, as a last thing, if you ever have to extract a large root, factor the exponent of the root first, if possible. For instance:

$$\sqrt[4]{N}=\sqrt[2]{\sqrt[2]{N}} \qquad \sqrt[6]{N}=\sqrt[2]{\sqrt[3]{N}}$$
$$\sqrt[8]{N}=\sqrt[2]{\sqrt[2]{\sqrt[2]{N}}} \qquad \sqrt[9]{N}=\sqrt[3]{\sqrt[3]{N}}$$

It is a lot easier this way. Try extracting the $\sqrt[12]{2}$. Then try extracting

$$\sqrt[2]{\sqrt[2]{\sqrt[3]{2}}}.$$

Which do you think is easier?

Letter to the editor

Dear Sir:

The November 1958 issue of THE MATHEMATICS TEACHER contained an article, "Teaching loci with wire and paint," by Donald A. Williamson, Bethesda-Chevy Chase Senior High School. The teachers in our mathematics department were interested in this article, as we are studying locus now in our plane geometry classes.

Our students made their own models of the nine locus theorems mentioned in the article. This can be done easily with construction paper. They used black paper for the given conditions and various colors for the loci. We feel that there is a definite gain in understanding this troublesome topic by having the students make their own models, which were used to illustrate compound loci also. These models were used later for a bulletin board display.

Sincerely yours,
Helen U. Toms
Mathematics Teacher
Greencastle-Antrim Joint School System
Greencastle, Pa.

Developing a city-wide algebra test

CAROL V. MCCAMMAN, *Calvin Coolidge High School, Washington, D. C.*

The story of how one city system developed a testing instrument to achieve greater uniformity in grading ninth-year algebra students.

SOME YEARS AGO our supervising director of mathematics was talking with the head of the teachers college in Mowbray, South Africa, about standards in mathematics. It was asked whether we had national examinations in mathematics; of course the answer was "No." Did we then have city-wide examinations? Again, "No." Well, then, just how were the mathematics standards set in the District of Columbia public schools? The director undertook to explain that the objectives were quite fully outlined in the course of study, that the teachers had to satisfy rather high requirements in regard to qualifications before they received their appointments, and hence were considered to be qualified to set their own standards. "Well," exclaimed her South African friend, "if that isn't an example of American free enterprise, I don't know what is!"

This "American-free-enterprise" system of every teacher for himself in regard to standards of achievement and grading was and still is practiced in Washington, but there has been mounting dissatisfaction with it. When it was desired to group students homogeneously in mathematics classes, the grade in the previous course was often used—but not always satisfactorily. While one could be confident that Miss Doe's student, with grade of *C*, had had a thorough course and could be expected to perform acceptably, Miss Roe's student might have a *B* or even an *A*, and yet would not have had all the topics listed in the course of study, and might not be expected to perform reliably in any of them. Students, too, were disturbed by differences in standards: after having had a teacher who believed that grades should be given with respect to the individual student's ability and that effort and application should count as much as achievement, it was hard for the student to find that in another teacher's class no amount of hard work would earn him a good grade unless he learned the subject. This was indeed a confusing state of affairs.

These differences in the beliefs and practices of mathematics teachers were aired at a joint meeting of junior and senior high school mathematics teachers in March, 1956. Discussions of grading and evaluation in mathematics showed great differences in point of view, particularly between junior and senior high school teachers.

To find out whether those who were most vocal at the meeting were also representative of teacher opinion, a committee prepared a questionnaire which was answered by 49 senior high and 109 junior high teachers. Responses to this questionnaire showed that the great majority of the teachers desired uniformity of standards and marks in all junior and senior high school subjects except remedial mathematics. This was especially conspicuous with regard to elementary algebra and the sequential mathematics in senior high; all but three or four of the teachers answering the questionnaire were in favor of uniformity of standards in these courses. An overwhelming majority of teachers recommended using city-wide or standardized tests, as well as class work and teacher tests, in determining a pupil's final mark.

There was a marked difference in the opinion of junior and senior high school teachers as to whether or not, in seventh and eighth grade mathematics and in ninth grade general mathematics, a pupil's mark should indicate his achievement in relation to innate ability, but both groups opposed marking in relation to ability in algebra and other sequential courses. About 80 per cent of senior high and 70 per cent of junior high teachers also opposed marking in relation to the achievement of others in the class. Approximately 3 to 1 favored uniform examinations in the same subject in elective courses, even though the ability of the pupils in the different classes varied. When asked whether a final mark should reflect such factors as interest, work habits, citizenship, appreciations, and character development, the senior high teachers were *against* it 3 to 1, while the junior high teachers were *in favor* of it 3 to 2.

On these and other points the questionnaire replies brought out the agreements and disagreements in regard to standards and marking. There was fairly general agreement that there should be more uniformity of standards and marking in the sequential courses. It was decided to start with elementary algebra, both because it was the first and basic course of the high school sequence and because both junior and senior high school teachers were almost unanimous in declaring that they desired uniformity in it. An algebra-test committee of 18 teachers was formed; alternate schools on an alphabetical list of junior and senior high schools selected a teacher, preferably an algebra teacher interested in testing, to be on this committee. The committee met with the mathematics supervisors, Mrs. Ethel Grubbs and Miss Veryl Schult, to implement the opinion expressed in the questionnaire replies.

The committee decided to have a city-wide algebra test, on an experimental basis, in the spring of 1957. What test should be used? Existing standardized tests were considered, but rejected—chiefly because they did not exactly fit the Washington course of study, or did not exist in enough forms to use year after year. It was suggested that the task of constructing a suitable test be turned over to the Educational Testing Service—but there was no money to pay for this. The committee was forced to the decision that it must devise its own test.

The test, it was felt, must be completely objective, and easy to administer and to score. The multiple-choice form was selected as meeting these requirements, and also because we felt that good mathematical items could be written in that form. Each committee member chose one section of the course of study, and wrote ten or twelve questions on it. A small committee selected 60 questions from those submitted, choosing a number on each topic in proportion to the number of weeks assigned to the topic in the course of study. The whole committee then met to take the proposed test, and afterward made further suggestions in regard to form, wording, difficulty, and so on.

While all of this was being done, the supervisors undertook to secure the approval and co-operation of the superintendents and of the principals. It was possible in the junior high schools to arrange to give the test to all elementary algebra pupils at the same time, in 90 consecutive minutes, on the same day. Administration of the test was arranged by the principal; in some cases it was administered by the assistant principal or counsellor, in other cases by mathematics teachers. In most of the senior highs such school-wide arrangements were not possible; in some the test had to be given half at a time, in the regular class period. The first city-wide test was taken by 1854 junior high pupils from 64 classes, and by 678 senior high pupils from 31 classes.

The committee felt that it was important to have a basis for comparing the teacher's rating of the pupil with the pupil's performance on the test. Hence

each algebra teacher was asked to send to the mathematics office a list of pupils showing the estimated letter grade as of May 10. After school on May 28, the day the test was given, the algebra teachers brought the answer sheets to the mathematics office. These were scored, using a punched-out key, checked, and recorded on the class lists that had been submitted earlier. A small committee worked far into the night tabulating the scores for each class and school, and computing percentiles. Thus it was possible the day after the test was given to send to the schools the scores of the individual pupils, and the percentiles for the city.

The city median for ninth grade algebra classes was 28 right out of a possible 60; scores ranged from 5 to 59. Medians for the classes of different teachers ranged from as low as 14 to 38. While great variation in class medians was to be expected since most of the junior highs use homogeneous grouping, it was found that in most classes teachers gave the full range of letter grades regardless of the ability of the group (or perhaps no *F*'s in a very good class or no *A*'s in an extremely poor one). Thus, although in replying to the questionnaire the teachers said algebra students should *not* be graded in relation to the rest of the class, in practice most often they *were* being so graded.

Study of the class lists, with the teacher's grade estimated prior to the test compared with the test score, showed some extreme differences in grading. The students rated *A* and *B* in some classes had poorer scores on the test than the *D* and *F* students in certain other classes. In one school with two algebra classes, the *D* students in one class made better test scores than the *A* students in the other class!

From distributions made showing the tentative letter grades and the test scores, we had, for each teacher, the range of scores and the median score for each letter grade. For example, Teacher X, with 27 students, gave one *A*; this student made 53 on the test. The two *B* students scored 40 and 48, with a median of 44. The 11 *C* students scored from 19 to 39, with a median of 32. The 4 *D* students scored from 16 to 36, with a median of 26. The 2 *F* students scored 19 and 30, with a median of 24. Thus, while there was of course much overlapping, there was a steady improvement of test scores from the lowest to the highest assigned letter grades. This was not true in all cases, however; in some cases there was almost no relation between the student's letter grade and the test score. I am not talking now about individual discrepancies, which of course will always occur, but about *all* the pupils assigned a certain letter grade. In one class, for example, the *A* students, the *C* students, and the *F* students all had the same median score, 20. In such a class it was clear that, on whatever basis the letter grade was assigned, it was something quite different from the kind of knowledge called for by the test.

Perhaps I should say that the figures I have quoted have been for the ninth-grade algebra students. The senior high school classes in elementary algebra were composed largely of students who had not been considered qualified to take algebra in the ninth grade; in many cases they were not considered good prospects for algebra even later in school, but were taking it at parental insistence. The senior high school median was 20, eight points below the junior high median. The teachers of the senior high algebra classes were urged to use the junior high percentiles in interpreting their pupils' scores, since it was felt that standards of achievement in elementary algebra, when taken in senior high, should certainly not be lower than when taken in the ninth grade.

All this information—the range and median test score for each school, for each teacher, and for each letter grade for each teacher—was given to the algebra teachers in the fall following the first test, and was subsequently discussed at a meeting of all junior and senior high

school mathematics teachers. The information was coded in such a manner that no teacher or school was identified except the individual teacher receiving the information.

In general, reaction to the test was good; teachers felt that it was very helpful. The scores, interpreted in the light of city-wide percentiles, could be used, along with other information, in making the pupil's final grade. These scores were an objective measure which pupils, parents, and principals could understand. Especially when an algebra class was much above or much below average, the test scores justified giving grades that were very different from normal in distribution. Some teachers reported that they found that the test provided good motivation, especially for review, and that students were very much interested in their own scores and how, as individuals and as a class, they compared with city percentiles. The mathematics supervisors observed other benefits during the following school year; some principals, for example, were selecting pupils for algebra much more carefully. There was also closer adherence by the teachers to the course of study than in the previous year, when one teacher had spent six weeks on statistical graphs, and some others had covered little more than half of the prescribed course.

For efficiency in the early stages of this program, it was decided to have the same committee continue during 1957–58, the second year of the program, although in the future it is planned to change about half the members of the committee each year. The committee made a shorter test on the first half of the course, to be used in the regular class period at the end of the first semester. This test would serve to give students practice in the type of test to be used at the end of the year, and would let them and their teachers see how they stood at that point. Correction was handled in the same way as before, with the individual pupil scores, the school median, and the city percentiles being returned to the school the day after the test was given. In addition, the committee suggested a letter-grade interpretation of the test scores, and made the following recommendation: "In general, it seems reasonable that a grade for any pupil should not vary more than one letter grade" from the grade on the test. While this midyear test had no official status, many of the teachers did follow the committee recommendation in regard to using it in grading. In one school, the January test had an unexpected result: the performance of an algebra class on the test was so extremely low that the principal reclassified the whole group as general mathematics, with consequent change in the course of study for the rest of the year.

The construction of the second city-wide test and its administration in May of this year followed much the same pattern as the year before, except that the experience we had gained made possible a better organization of the correction, checking, and tabulation of results. Again, the individual test scores, the school median, the city percentiles, and a recommended interpretation of grades were all sent to the schools the day after the test was given. Two weeks later a more detailed report was made both to the algebra teachers and to the principals, showing in addition the median and range of scores for each school, and the median for each class. Again, the report was coded, with the key given only for the particular school. Since then, the range and median test score for each of the estimated letter grades have been obtained for each class.

How do the second-year results compare with the first? No statistical procedures or pretesting were used to equate the two tests, but it was the intention of the committee of experienced teachers who constructed the second test to make it comparable to the first. Hence it was felt that the city junior high median of 36 the second year represented a real gain in achievement over the 28 the year before. Of course not all the gain was due to either

better teaching or better learning; part of the gain was because some of the least able students had been directed out of the algebra classes; part was because teachers and students were more familiar with the type of test, and the kind of algebra competencies that the committee expected of students.

The second year there was much better correspondence between the teacher's estimated grade and the test score. This was evident in individual classes, and in the fact that median test scores in different classes for the same letter grade did not vary so extremely as they had the first year.

Committee discussion of the test, conversations about it between individual teachers or principals and the supervisors, and the comments from the floor following the panel discussion of the test and its results all seemed to indicate general approval of the idea of a city-wide test, and of the way in which we were proceeding. A small but very vocal minority were strongly of the opinion that such a test should not be teacher-constructed, -scored, or -analyzed, feeling that these things should be done by an outside agency, qualified and impartial. At present this is impossible, since there is no such staff within our school system, nor money to pay to have it done outside. Even if it were financially possible, it is a question whether most teachers would prefer such a system to our own, which permits us to give the test late in the semester since we have the results ready the next day. A few teachers have objected strenuously to any recommendation from the committee or anyone else in regard to interpretation of the letter-grade value of the test scores, and especially to the recommendation of the use of the test scores in grading pupils (i.e., that the pupil's grade, in general, differ by no more than one letter grade from the test score, so that a student who earned *C* on the test might get *B*, *C*, or *D* in the course, but not *A* or *F*). This objection was made despite the fact that this was a recommendation, not a requirement, and made with the realization that, even if it were followed, there might be individual exceptions. Most teachers, however, wanted interpretation of the test scores, and seemed to find that the recommendation as to their use was a reasonable one, and if followed would contribute to a desirable uniformity of standards.

The general reaction to the algebra-test program has been such that we are planning to continue it—for the present, at least—in pretty much the same fashion as last year. The question of extending the program to geometry and intermediate algebra has been raised, but not yet decided. However, some of the high schools have been experimenting with school-wide tests in these subjects, using in some cases standardized tests and in others teacher-made tests. Other departments have shown interest, too, and our high school teachers' association held a panel discussion on the pros and cons of a city-wide system of examinations.

Like most things, our program involved work, but—we thought—not too much. For the teacher in general, it meant relief from the need of constructing or procuring a final examination in algebra, and instead of correcting his own algebra test papers he went to a central place and spent about an hour and a half correcting someone else's papers. The committee members of course spent additional time, but not more than they might have spent on some other department committee. Further, many of them felt that being on the committee was a very valuable experience—discussion of the construction of the test, and of the difficulty, suitability, form, and foils of the test items was interesting and illuminating to all concerned. The four people who did a great deal of additional work (the two supervisors, the editorial committee chairman, Mr. Anselm Fisher, and his technical assistant) were so convinced of the value of the program that the time it took was given gladly.

If you are not already participating in a co-operative testing program with other

teachers, I recommend that you begin soon. In my experience, the resulting discussion of objectives, standards, and methods is of great value, in addition to the value of the joint-testing program. Such a venture need not be city-wide or even school-wide; it requires only two or more teachers who wish to co-operate. It can be as simple or as elaborate as circumstances warrant.

One caution, though. It is not only necessary to construct or choose your test with care, but it is also essential that the security of the test be protected until it is given, that it be administered under standard conditions to all pupils involved, that there be a uniform correction procedure, and that there be some kind of uniformity in handling the use of the test results after it has been corrected. If some of the questions are not objective, a grading procedure should be worked out so that any particular question is scored uniformly on all papers.

As you have probably gathered, I believe that system-wide tests are valuable. In a place such as Washington, where there is no provision for such tests, I believe that an approach such as ours is a good one: first getting the opinions of the mathematics teachers, then having representative teachers work out a test to be used experimentally, and finally requesting and using reactions and contributions of teachers. I think we have made a good beginning with our algebra test, and I am hopeful that we shall continue and extend our program.*

* This article was written for the August, 1958, NCTM Summer Meeting. Since that time, final examinations have been the subject of much discussion in Washington, D. C., and a committee of principals, supervisors, and teachers is preparing to make recommendations to the superintendent.

Have you read?

BALL, ROGER E., and GILBERT, ALLAN A., "How to Quantify Decision Making," *Business Horizons*, Winter 1958, pp. 73–79.

This article presents an interesting application of mathematics in an area where at first glance it would appear impossible. What problems of decision can be solved by mathematics? Would you expect that such problems as mergers, manufacture of new products, expansion, and the development of new markets could be solved by mathematics? This article tells how you can use a matrix, and gives an excellent illustration. It deals with available alternates, future conditions, levels of production, and so forth. How does the matrix help? It takes some of the uncertainty out of decision-making and provides for logical, orderly considerations, common measures, equality of considerations, and recognition of all possible alternates. This will interest not only you but your students.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

MULLER, CORNELIUS H. "Science and Philosophy of the Community Concept," *American Scientist*, September 1958, pp. 294–308.

This is not a mathematics article, but it does a splendid service in relating the community to scientific analysis. It points out, for example, the infinite series of arrangements possible in the community and the relation of the total to the sum of its integral parts. Does this mean the community is fully determined? Muller relates the beginning of the community to the beginnings of biological life and the part played by environment and variation.

Psychological as well as biological communities play their part in the total structure.

What does all this lead toward? Perhaps to the point of view that you cannot define a community, that organism and community are not comparable. Does this mean there is no community? Read this and decide.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

EDITORIAL NOTE: Of great importance in the history and development of mathematics have been the International Congresses of Mathematicians, the first of which was held in Zurich, Switzerland, in 1897. Prior to this meeting there had been two other international gatherings of mathematicians, the *Congrès international de bibliographie des sciences mathématiques*, held in Paris in 1889, and the International Mathematical Congress, held in connection with the World's Columbian Exposition at Chicago in 1893.

At the Zurich Congress of 1897 the aims of the congress were formulated as: (1) to promote personal relations among the mathematicians of different countries, (2) to give a survey of the present state of various mathematical branches and to offer an occasion to treat certain questions of special importance, (3) to deliberate and to decide on the problems and the organization of future congresses, and (4) to discuss questions of bibliography, terminology, and so forth, in regard to which an international understanding seems necessary. This first congress numbered 242 members, representing 16 different countries. It was decided at this congress to hold future international congresses in various countries at intervals of from 3 to 5 years, and, except for an interruption during the war years, this plan has been followed.

This past summer the International Congress was held at Edinburgh, Scotland. For the benefit of those of us who were unable to attend the congress, the editor invited Professor Guggenbuhl to write a brief descriptive account of the affair.

International Congress of Mathematicians, Edinburgh, 1958

by Laura Guggenbuhl, Hunter College, New York, New York

As we drove across the St. George bridge and up the hill, a large ICM sign beckoned to us and told us that we were going in the right direction, for we were on our way to register for the International Congress of Mathematicians in Edinburgh. The Congress was held under the patronage of H.R.H. Prince Philip, the Duke of Edinburgh, and at the invitation of the City and University of Edinburgh and the Royal Societies of London and Edinburgh. It would not open until the following day, but headquarters in the University Union was already a beehive of activity, and on all sides one heard a warm exchange of greetings. Four years seemed suddenly to have vanished, as one met friends from home and abroad whom one had not seen since the last congress in Amsterdam.

Upon registration each member received, with the compliments of British European Airways Corporation and British Overseas Airways Corporation, a packet containing Congress material, a Con-

gress number and letter box. The packet included a handsome Congress badge depicting Edinburgh Castle, a yellow booklet entitled *Congress Guide*, a green booklet giving a list of members, and a red booklet with *Abstracts of Short Communications and Scientific Programme*. Each member also received a beautifully illustrated book, *Presenting Edinburgh*, from the publishers Messrs. Oliver and Boyd, Ltd., and other pamphlets. A problem presented itself immediately, even before the Congress began. Should one retire at once to the serene quiet and Victorian comfort of the ivy-covered walls of the Roxburghe Hotel, where several American delegates were so pleasantly housed, to read this material? The problem was solved by the decision to attend the informal social gathering which had been planned for the evening in the University Union.

At the opening session in the McEwan Hall on Thursday, August 14, Fields medals[1] were presented to R. Thom of Strasbourg for work in topology, and K. F. Roth of London for work in number theory. H. Hopf of the Federal Institute of Technology in Zurich and H. Davenport of London then addressed the Congress on the subjects of the work for which the Fields medals had been awarded.

The scientific program of the Congress was divided into the following eight sections:

I	Logic and Foundations
IIA	Algebra
IIB	Theory of Numbers
IIIA	Classical Analysis
IIIB	Functional Analysis
IV	Topology
VA	Algebraic Geometry
VB	Differential Geometry
VI	Probability and Statistics
VIIA	Applied Mathematics
VIIB	Mathematical Physics
VIIC	Numerical Analysis
VIII	History and Education

[1] At the International Congress held in Toronto in 1924, it was decided that at each Congress two gold medals should be awarded, these medals to be known as the Fields medals in honor of Professor J. C. Fields, the Secretary of the 1924 Congress, who presented the Congress with a fund to subsidize the medals. The first Fields medals were awarded at the congress held in Oslo in 1936.

Each busy day that followed began with a one-hour address on a general subject, given at the invitation of the Organizing Committee. In sectional meetings later in the day some half-hour lectures were given at the invitation of the Organizing Committee, and quarter-hour sessions were reserved for contributed papers. The *Congress Guide* gave an outline program for the entire Congress in each of four languages, English, French, German, and Russian; a schedule of the invited lectures; an alphabetical index of the names of speakers; and a map of the Congress grounds. Thus it was a simple matter to locate the section, time, and place where a particular member would speak. However, since several lectures one would want to hear were frequently given simultaneously in different buildings, it was utterly impossible for anyone to hear all speakers. A brief summary of some of the above information appears in Tables 1 and 2 on pages 192 and 193.

No doubt the reader of THE MATHEMATICS TEACHER will be particularly interested in the activity of Section VIII. As in other sections, there was an invited address, the lecture of J. E. Hofmann (Germany) titled "Über eine Euklid-Bearbeitung des Albertus Magnus." Other speakers and papers at Section VIII were the following:

Bunt, L. N. H. (Netherlands): An investigation into the possibility of teaching probability and statistics in Dutch secondary schools.

Burckhardt, J. J. (Switzerland): Zwei griechische Ephemeriden.

Calvert, H. R. (England): The need for the sector as a computing and drawing instrument.

Cassina, U. (Italy): A study of the present state of teaching the elements of geometry in Italy.

Drenckhahn, F. (Germany): Elementargeometrie im Unterricht: vom Gestaltlichen aus und in didaktischen Experimenten.

Eisele-Halpern, C. (U.S.A.): The theory of probability in Charles S. Peirce's logic and history of science.

May, K. O. (U.S.A.): Stimulating undergraduate research.

McConnell, J. (Eire): Sir Edmund Whittaker's philosophy of science.

Ness, W. (Germany): Beispiel und Gegenbeispiel in der Mathematik.

Piene, K. (Norway): Statistics in secondary schools.

Storer, W. O. (England): Symbolism and the rules of operations in school mathematics.

TABLE 1

ADDRESSES GIVEN AT THE INVITATION OF THE ORGANIZING COMMITTEE[2]

9:30–10:30		2:30–3:30
Akhiezer (U.S.S.R.) The classical moment problem and its continuous analogues. Feller (U.S.A.) On some new connections between probability and classical analysis. Hirzebruch (Germany) Komplexe Mannigfaltigkeiten.	Friday August 15	Grothendieck (France) The cohomology theory of abstract algebraic varieties. Schiffer (Netherlands) Extremum problems and variational methods in conformal mapping. Uhlenbeck (U.S.A.) Some fundamental problems in statistical physics.
Eilenberg (U.S.A.) Applications of homological algebra in topology. Lanczos (Eire) Extended boundary value problems. Siegel (Germany) Ideen und Probleme der Zahlentheorie.	Saturday August 16	
Cartan (France) Sur la théorie des fonctions analytiques de plusieurs variables. Roth (England) (Fields Medalist) Rational approximations to algebraic numbers.	Monday August 18	Steenrod (U.S.A.) Cohomology operations and symmetric products.
Pontryagin (U.S.S.R.) Optimal processes of regulation. Thom (France) (Fields Medalist) Global differential geometry and function spaces. Wielandt (Germany) Entwicklungslinien in der Strukturtheorie der endlichen Gruppen.	Tuesday August 19	
Kleene (U.S.A.) Mathematical logic: constructive and nonconstructive operations.	Wednesday August 20	
Alexandrov (U.S.S.R.) Differential geometry in the large and metric methods in differential geometry. Chevalley (France) Sur la théorie des groupes algèbriques. Gärding (Sweden) Trends and problems in partial differential equations.	Thursday August 21	

TABLE 2

SECTIONAL ADDRESSES PRESENTED AT THE INVITATION OF THE ORGANIZING COMMITTEE[2]

	Friday August 15	Saturday August 16	Monday August 18	Tuesday August 19	Wednesday August 20	Thursday August 21	
I				Kreisel (England) Beth (Netherlands)	Markoff (U.S.S.R.)		I
IIA	Cernikov (U.S.S.R.)		Higman (England)				IIA
IIB	Deuring (Germany)		Roquette (Germany) Shimura (France)				IIB
IIIA	Arnold (U.S.S.R.) Grauert (Germany)	Heins (U.S.A.) Bers (U.S.A.)					IIIA
IIIB			Minakshi Sundaram (India)	Lions (France) Krein (U.S.S.R.)	Menchoff (U.S.S.R.)		IIIB
IV			Papakyriakopoulos (U.S.A.) Kosiński (Poland)		Moore (U.S.A.) Bott (U.S.A.)	Wu Wen-tsün (China)	IV
VA		Matsusaka (U.S.A.) Samuel (France)	Nagata (U.S.A.)	Segre (Italy)			VA
VB	Chern (U.S.A.)	Milnor (England)		Wang (U.S.A.)		Nijenhuis (U.S.A.)	VB
VI			Chung (U.S.A.) Rényi (Hungary)		Gnedenko (U.S.S.R.) Savage (U.S.A.)		VI
VIIC		Lehmer (U.S.A.)	Wijngaarden (Netherlands)		Rutishauser (Switzerland)		VIIC
VIII	I.C.M.I.	Hofmann (Germany) I.C.M.I.	I.C.M.I.				VIII

A number of unusually interesting and discussion-provoking sessions were arranged for Section VIII by the International Commission on Mathematical Instruction[3] (I.C.M.I.). On Friday, Saturday, and Monday, a topic in the teaching of mathematics was introduced in the morning session by a General Reporter. In the afternoon additional remarks were made by representatives of national subcommittees, and then these remarks were followed by a general discussion open to all present. The topics, reporters, and chairmen for these sessions were the following:

1. Mathematical instruction up to the age of fifteen.
   General Reporter—H. Fehr (U.S.A.)
   Chairman, Morning—H. Behnke (Germany)
   Afternoon—W. Servais (Belgium)
2. The scientific foundation of mathematics on the secondary school level.
   General Reporter—H. Behnke (Germany)
   Chairman, Morning—H. Fehr (U.S.A.)
   Afternoon—L. N. H. Bunt (Netherlands)
3. Comparative studies of methods of teaching beginning geometry.
   General Reporter—H. Freudenthal (Netherlands)
   Chairman, Morning—J. Desforge (France)
   Afternoon—G. Kurepa (Yugoslavia)

[2] Names are omitted if not included in the List of Members, or if title of address is missing.

[3] The important International Commission on Mathematical Instruction was inaugurated at the fourth International Congress, held in Rome in 1908.

Although three languages, English, French, and German, were used for the presentation of these reports and the resulting discussion, remarks made in French and German were summarized in English by the chairman.

During the Wednesday morning session of Section VIII, a group of well-known Americans, nominated by the United States Committee on Mathematical Instruction, spoke on their fields of special interest in the teaching of mathematics. In this group were:

A. W. Tucker: The work of the (American) Commission on Mathematics.

H. E. Vaughan: The Illinois experiment in high school mathematics.

W. L. Duren, Jr.: The reform of college mathematics teaching in the United States.

G. Baley Price: Institutes for mathematics teachers.

G. B. Allendoerfer: Teaching mathematics on television.

A Scottish school inspector was heard to remark that he was so relieved to see that the Americans had finally come to some of these reforms and were now doing some of the things which have been done for many years past in various countries in Europe. But the supreme test of a stimulating and provocative three-language discussion was well met by the I.C.M.I. program. In passing, one might note that the only Russian, listed as a speaker among the representatives of the national subcommittees, did not appear. Also, some listeners were not entirely convinced by the claims of the Yugoslavian representatives who insisted that they taught group theory to ten-year-old children.

Turning now to the social program of the Congress allows one to say, with the greatest possible pleasure, that here there was every evidence of the same thoughtful, thorough, devoted, and quiet planning which made the scientific program such an outstanding success. On Thursday afternoon, August 14, after the opening ceremonies, all members and associates were invited by the Lord Provost, Magistrates, and Council of the City and Royal Burgh of Edinburgh to a garden party at Lauriston Castle. The *Congress Guide* gives the following description of this enchanting place at the water's edge:

> Lauriston Castle is typically Edinburgh in that it is a beautiful place backed by historical association. As seen today it is in two parts. The older was built by Sir Archibald Napier of Merchiston, whose son, John, invented logarithms. The newer part was added by a banker friend of Sir Walter Scott, after 1823.
>
> Between these periods Lauriston Castle was owned for 140 years (1683–1823) by the Law family. John Law of Lauriston achieved international fame as financial adviser to the French throne, but never lived there.
>
> The latest private owner was Mr. William Robert Reid, who bequeathed, after the death of his wife, the Castle to the nation, for whom the Edinburgh Corporation are trustees. Mr. Reid was noted as a collector of antiques, of which the Castle is full of beautiful examples.

An endless supply of most delicious refreshments served in large tents which had been set up at various points on the spacious lawns, music by a marching highland band, and a warm sun made this event a memorable occasion for the 2000 persons who attended the party.

On Sunday, August 17, the members of the Congress embarked from Glasgow on two separate steamers for an all-day cruise on the Clyde River. Particularly interesting spots along the river were the docks, the shipyards where the giant ocean liners S. S. "Queen Mary" and S. S. "Queen Elizabeth" were built, and the green hills and pleasant vacation resorts along the banks of the river. A warm sun produced the high lights and shadows which were the delight of photographers. The luncheon and high tea served on the steamer and the many opportunities for exchange of greetings were enjoyed by all.

Many additional excursions (with a separate set for associate members) had been planned. There were sight-seeing tours of the City, a bus trip to the Trossachs, one to nearby abbeys, a trip to St. Mary's Loch, and another to the famous Forth Bridge. Visits were arranged to a

coal mine, a hydroelectric plant, a newspaper firm, a department store, factories, workshops, the Royal Botanic Garden, and the Zoological Park. For the evenings there were dancing, concerts, films, and a Reception in the Royal Scottish Museum. The *Congress Guide* also spoke about the ruins of Merchiston Castle, where John Napier was born, and the nearby tomb of Colin Maclaurin.[4]

Lest the reader fear that Congress members were overwhelmed by the great activity described above, one must hasten to add a few words about the delightful exhibition of books which was arranged for the occasion. Indeed, a member could easily have withdrawn for the entire week to the quiet enjoyment of these books, without venturing beyond the walls of the National and University Libraries and the other places where these exhibits were housed. At Messrs. James Thin, Booksellers, an exhibit of about 700 mathematical books from many countries was arranged by T. A. A. Broadbent (England), with the assistance of C. B. Allendoerfer (U.S A.), E. Bompiani (Italy), G. Choquet (France), L. Locher-Ernst (Switzerland), E. Kamke (Germany), and Mr. Ainslie Thin. Another exhibition of very great interest, devoted to school textbooks and located in Moray House, where the meetings of Section VIII were held, was arranged by the International Commission on Mathematical Instruction. The latter exhibition, which has been prepared by the countries supporting the work of the Commission, is permanently housed in Paris.

About 2400 persons from all parts of the world attended the Congress. The American delegation numbered about 350 members and 200 associates. The languages of the Congress were English, French, German, and Russian, but many speakers spoke in English even though it was not their native tongue. All abstracts printed in Russian were accompanied by an English translation. Members of the Organizing Committee seemed to prefer to remain in the background, nor were their names listed in the *Congress Guide*. Some of those who brought greetings spoke of the fact that, when the Congress was first planned for Edinburgh, it was hoped that Scotland's beloved late Sir Edmund Whittaker, who died on March 24, 1956, would be the president.

In a recent letter the Congress Secretary has graciously supplied the following information. The principal officers of the Congress were as follows:

Chairman of the Congress Committee:

The Rt. Hon. Ian Johnson-Gilbert, Lord Provost of the City of Edinburgh;

Vice-Chairman: Sir Edward V. Appleton, Vice-Chancellor and Principal of the University of Edinburgh;

President of the Congress: Professor W. V. D. Hodge;

Secretary of the Congress: Dr. F. Smithies;

Treasurer of the Congress: Mr. A. L. Imrie, City Chamberlain of the City of Edinburgh;

Local Secretary: Dr. R. Schlapp.

Chairmen of subcommittees were as follows:

Accommodation: Mr. C. H. Stewart, Secretary of the University of Edinburgh;

Entertainments: Mr. D. M. Weatherstone;

Excursions: Dr. D. E. Rutherford;

Finance: Mr. J. G. Dunbar, Treasurer of the City of Edinburgh;

Proceedings: Dr. L. J. A. Todd;

Scientific Programme: Professor M. H. A. Newman;

Registration and Reception: Professor R. A. Rankin;

Travel Grants: Professor A. G. Walker;

Book Exhibition: Professor T. A. A. Broadbent.

[4] Colin Maclaurin (1680–1751) was one of the ablest mathematicians of the eighteenth century.

Officers of the Executive Committee were:

Chairman: Professor W. V. D. Hodge;
Vice-Chairman: Professor A. C. Aitken;
Secretary: Dr. R. Schlapp;
Advisory Secretary: Mr. J. MacPherson.

A short closing session, with Professor W. V. D. Hodge, Professor H. Hopf, Dr. F. Smithies, and Professor B. Jessen on the platform, was held in the McEwan Hall on Thursday afternoon, August 21, at 2:30 P.M. At this meeting, Professor Jessen made the speech of thanks on behalf of the members of the Congress. Professor Hopf reported on the meeting at St. Andrews of the Assembly of the International Mathematical Union. Professor Hopf also reported that further deliberations and consultations would be needed before an announcement could be made regarding the location of the next Congress to be held four years hence.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

THIRTY-SEVENTH ANNUAL MEETING

April 1–4, 1959
Baker Hotel, Dallas, Texas
Arthur W. Harris, 4701 Cole Avenue, Dallas 5, Texas

JOINT MEETING WITH NEA

July 1, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

Other professional dates

Mathematics Section of the New York Society for the Experimental Study of Education

March 14; April 17, 1959
Teachers College, Columbia University, New York, New York
John A. Schumaker, Montclair State College, Montclair, New Jersey

Illinois Council of Teachers of Mathematics

March 28, 1959, Macomb, Illinois
April 11, 1959, Normal Illinois
April 15, 1959, Charleston, Illinois
April 18, 1959, Arlington Heights, Illinois
April 18, 1959, Carbondale, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Association of Mathematics Teachers of New York State

May 1–2, 1959
Syracuse, New York
George Lenchner, Valley Stream North High School, 750 Herman Avenue, Franklin Square, New York

Tenth Annual Conference of the Michigan Council of Teachers of Mathematics

May 1–3, 1959
MEA Camp, St. Mary's Lake, Battle Creek, Michigan
Elizabeth N. Scott, Emerson Junior High School, Flint, Michigan

Chicago Elementary Teachers' Mathematics Club

May 11, 1959
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Romana H. Goldblatt, Burley School, Chicago, Illinois

Fortieth Summer Meeting, Mathematical Association of America

August 31–September 3, 1959
University of Utah, Salt Lake City, Utah
Harry M. Gehman, University of Buffalo, Buffalo 14, New York

Edited by Lucien B. Kinney, Stanford University, and
Dan Dawson, Stanford University, Stanford, California

Stimulating interest in junior high mathematics

by Margaret F. Willerding, San Diego State College,
San Diego, California

Introduction

The job of the junior high school mathematics teacher is a very difficult one. His students range in achievement from primary level through senior high level and above. To keep all his students interested—and the main task of any teacher is to make his subject interesting—the junior high mathematics teacher must have materials at hand for all of these achievement levels.

It is not difficult to have materials for all levels of achievement. The difficult task is to have third-grade material that the slow junior high student will think worthy of his maturity, to know the answers to the questions that the gifted student asks, and to stimulate the gifted student to continue his studies of mathematics and the sciences.

With students in the low achievement group the teacher will be concerned with such problems as:

1. teaching multiplication when the student does not know the addition facts;
2. teaching division when the student does not know the multiplication facts;
3. teaching percentage when the student does not fully understand the rational operations.

It is a well-known fact that a large majority of students do not like mathematics. Perhaps one of the reasons for this dislike is that they do not fully understand the basic concepts of mathematics. We never like something we cannot understand.

Another big problem facing the junior high mathematics teacher is how to teach those things which have been presented year after year since the third grade. In attacking this problem the first thing to remember is that the material must be presented in such a manner as to seem new and interesting.

The object of this paper is to present some methods and materials to make junior high mathematics interesting despite the fact that a great deal of the curricula is old stuff and has been presented many times.

Homemade slide rule

Among the slower group in a junior high mathematics class there are always a few who do not known the addition and subtraction facts. Because of this, these students have difficulty doing multiplication, division, and percentage problems. Since even the best mathematicians use calculators to do tedious arithmetic calculations, the junior high student will not feel it below him to use a slide rule, even a homemade one.

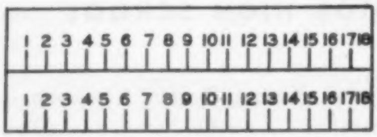


Figure 1

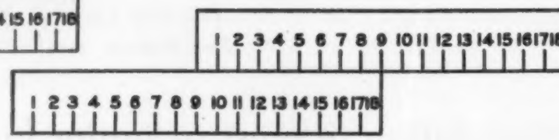


Figure 2

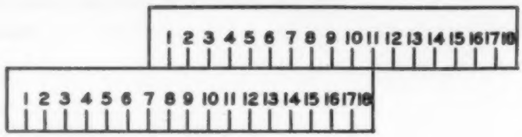


Figure 3

A very adequate slide rule for the basic addition and subtraction combinations can be made with two 18-inch rulers.

To construct such a slide rule, the only materials needed are two 18-inch rulers. Place them side by side as in Figure 1.

To add 9 to 5 on this slide rule, put the zero mark of the top ruler above the 9 mark on the bottom ruler as shown in Figure 2. Find the 5 mark on the top ruler. The answer to the addition problem is shown on the bottom ruler directly below the 5 mark on the top ruler.

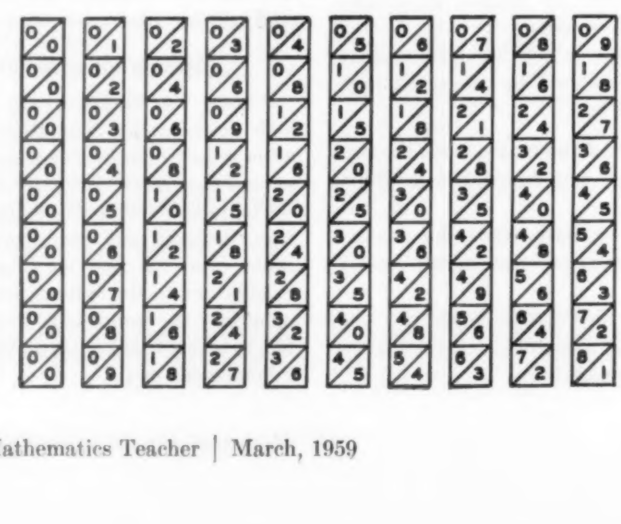
This simple slide rule can also be used to find the answers to all the basic subtraction combinations. To subtract 8 from 15, place the top ruler so that the 8 mark is directly over the 15 mark on the bottom ruler as shown in Figure 3. The answer 7 is found on the bottom ruler directly below the zero mark on the top ruler.

An extension of the use of this slide rule is the addition and subtraction of fractions and mixed numbers involving the so-called ruler fractions, those fractions whose denominators are 2, 4, 8, and 16.

Napier's Bones

A set of Napier's Bones can be constructed very simply on heavy cardboard, tongue depressors, or popsicle sticks. A set of the Bones is shown in Figure 4.

Figure 4



Napier's Bones are actually ten strips on which the products of the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, by one of the digits from 1 to 9 inclusive are given. Each bone is thus a multiplication table of the first ten numbers by a certain one-digit number.

To use Napier's Bones for multiplying 738 by 25, we pick out bones 7, 3, and 8 and arrange them as shown below in Figure 5.

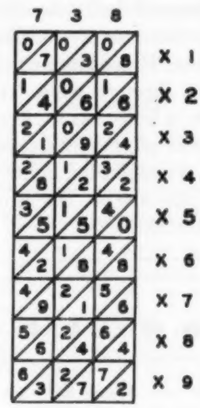


Figure 5

The product of 5×738 is given in the fifth row. The digits between the diagonal lines must be added to find the product. You read the product from right to left as shown in Figure 6.

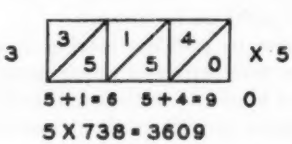


Figure 6

The product of 2×738 is found in the second row, as shown in Figure 7 at the top of the next column.

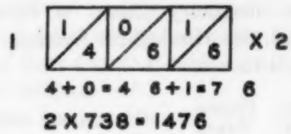


Figure 7

Since 2×738-1476, 20×738 = 14,760.

The product of 25×738 is the sum of the two products 5×738 and 20×738.

$$\begin{array}{r} 5\times738 = 3690 \\ \underline{20\times738 = 14760} \\ 25\times738 = 18456. \end{array}$$

The author has used Napier's Bones successfully with many seventh- and eighth-graders. Low and average groups have used them for multiplication and addition practice. Superior groups have used them in a unit on the slide rule and calculating machines. Interest in Napier's Bones stimulated superior members of the class to study other types of calculating machines and their history.

Making the diagonal lines of the bones red (or another color) and the horizontal lines black (a second color) sometimes facilitates learning to use Napier's Bones. This way the numbers to be added in the diagonals show up more clearly.

Hindu Arabic number system

All educators will agree that success in mathematics begins with a firm understanding of the Hindu Arabic number system. Building number systems of bases other than ten is a sure way to test students' understanding of the principles of the Hindu Arabic system.

To extend understanding of the Hindu Arabic number system, the author has used the "Hindogabic" System* (base 4). As a project the author took her students on an imaginary trip to Dogland located

* J. E. Eagle of San Diego State College created the Hindogabic Number System.

on the imaginary planet of Sirius. In Dogland, the Hindogabic System, shown below, is in use.

Hindu-arabic symbol	Hindo-gabic symbol	Hindogabic number name
0	0	zero
1	1	one
2	2	two
3	3	three
4	10	doggy
5	11	doggy-one
6	12	doggy-two
7	13	doggy-three
8	20	twoggy
9	21	twoggy-one
10	22	twoggy-two
11	23	twoggy-three
12	30	throggy
13	31	throggy-one
14	32	throggy-two
15	33	throggy-three
16	100	one houndred
17	101	one houndred one
18	102	one houndred two
19	103	one houndred three
20	110	one houndred doggy
21	111	one houndred doggy-one
22	112	one houndred doggy-two
23	113	one houndred doggy-three
24	120	one houndred twoggy

While on their imaginary trip to Dogland, students learned to count, to write number symbols, to add and to multipy in the Hindogabic system. Better students learned also to subtract and divide. Interest was so high during this project that many students created number systems of other bases, such as 8 and 6.

With a small group of better students the binary system was studied. Its history and an application are interesting units for study.

The binary system (base 2) needs only two symbols, 0 and 1, for writing numbers. To learn the places of the binary system an open-end abacus was used.

Since the base of the binary system is 2, the students quickly saw that when a number was represented on the abacus,

Figure 8
Open-end abacus showing 1101 (binary system)

each rod had either one bead on it or no beads on it.

After the binary system was understood, a simple device using a piece of board 24 in.×4 in.×1 in. and a strand of Christmas-tree lights (parallel circuit so that each bulb burned individually) was constructed to represent numbers in the binary system. Holes large enough for the bulbs to slip through were drilled in the board. When a bulb was lit, it represented 1; when it was off, it represented 0. Thus, in Figure 9, if the shaded lights are off and the white lights are on, the number represented is 1,001,001.

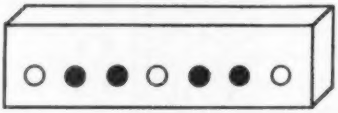


Figure 9
1,001,001 (binary system)

Problem solving

Statement problems often cause junior high students much trouble. Pages of problems in arithmetic texts look very formidable to students and often do not hold their attention. Problem situations from the life and experiences of the child are much more interesting to him than printed problems in the text.

Interest in problem solving is stimulated by bulletin-board problems of many different kinds. Advertisements from daily newspapers furnish excellent material for units on problem solving. Posters can present the arithmetic in the daily papers

Places of binary system

2×2×2×2×2	2×2×2×2's	2×2×2's	2×2's	2's	1's
32's	16's	8's	4's	2's	1's

effectively. Each poster may contain problems that involve several arithmetic operations. Too often a problem set in the text is constructed so that all the problems can be solved by the same basic method. Students quickly sense this, and so not *think* as much as they should. In making "problem sets," the instructor can use one advertisement to make problems involving several operations.

Using the teacher-made problem sets as a guide, students can make their own problem sets. Having the class give constructive criticism of student-made problem sets leads to some very interesting and worthwhile discussion.

Conclusion

The ideas presented in this paper are not necessarily new or original. The author presents them here to stimulate teachers of junior high mathematics to think up ideas of their own that will create interest in their subject.

In his immortal work, *The Prophet*, Gibran says, on teaching:

> No man can reveal to you aught but that which already lies half asleep in the dawning of your knowledge.
>
> The teacher who walks in the shadow of the temple, among his followers, gives not of his wisdom but rather of his faith and lovingness.
>
> If he is indeed wise he does not bid you enter the house of his wisdom, but rather leads you to the threshold of your own mind.

The objective of this paper is to lead you the reader to the threshold of your own ideas so that you will find new ways to stimulate interest and enthusiasm for mathematics among your junior high students.

Solutions to problems in *The Mathematics Student Journal* of March 1959

The game. The second player can be sure of winning. He should move in such a way that the black and white pieces are separated by equal distances. For example, if White moves from 9 to 12 as his first move, there are three spaces between White and Black on the lower row, namely the squares 13, 14, 15. Black moves from 8 to 5, and this leaves also three spaces, squares 2, 3, 4, between them on the upper row. *Black need never retreat.* After each move of White, Black observes on which row there is the longer distance between the pieces; he moves forward to reduce this distance, and make it equal to the distance on the other row.

Surprisse in Shorthand. Examples 3 and 4 contain statements true for every number you may think of. Example 5 is untrue for every number except zero. Example 6; statement (ii) is true for every whole number and for every fraction.

Sums in Code.

5	999	89	23
5	1	9	23
5	——	—	23
—	1000	98	23
15			—
			92

In the last of these, four times *ON* makes *GO*. *GO* must represent a number less than 100, so *ON* must be less than 25. So *O* is either 1 or 2. If *O* were 1, *GO* would be an odd number, and could not be 4 times *ON*. So *O* must be 2. *GO* must therefore be either 82 or 92. But 82 does not divide by 4. So *GO* must be 92.

Problem 131. It is easily seen that $a^2-b^2-bc+ac=0$. That is, $(a-b)(a+b)+c(a-b)=0$. So $(a-b)(a+b+c)=0$. By a similar argument we find $(b-c)(a+b+c)=0$ and $(c-a)(a+b+c)=0$. If $a+b+c$ is not zero, we can divide by it in each equation, and hence find $a=b=c$. But we were told that a, b, c were *not* all equal. So we must have been wrong in supposing $a+b+c$ not to be zero. Hence $a+b+c=0$.

Problem 133.

	If the window was broken by John	Henry	Thomas	David	Ernest
the statement made by John was	False	True	True	False	False
Henry was	True	False	True	True	False
Thomas was	False	False	False	False	True
David was	True	True	False	True	False
Ernest was	False	False	True	False	True

We are assured that three of the sons always speak the truth. There are three true statements only if Thomas broke the window, so he must have done it.

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

During the last several years much has been said about the need for providing more adequate programs for the gifted student. The article by Mr. Jackson which follows is evidence that these suggestions have been put into practice in at least one large school system. The actions that the Minneapolis schools have taken are probably typical of what is being done in many schools. Although these activities took place in a large school system, the report contains ideas that any school can use to make provision for the students gifted in mathematics.

The superior pupil in mathematics

by Harvey O. Jackson, Minneapolis Public Schools, Minneapolis, Minnesota

This report is a summary of the information obtained from the chairmen of the mathematics departments of the Minneapolis junior and senior high schools in response to a questionnaire sent to them in April, 1958.

Each mathematics chairman was asked to answer these three questions for his school:

What provisions are made for *identifying* the superior pupil?
What provisions are made for *enriching* the program of the superior pupil?
What provisions are made for *accelerating* the program of the superior pupil?

For the purposes of this report the superior pupil was described as one to whom *most* of the following statements would apply:

He obtains an IQ score of 120 or above on a group test of mental ability.
He attains a percentile rank of 90 or better on standard mathematics achievement tests.
He has A marks in previously-taken mathematics courses.
He shows strong interest in learning mathematics.
He has good work and study habits.

1. *What provisions are made for identifying the superior pupil?*

Reports from both the junior and senior high schools indicate that the most important items of information for use in identifying superior pupils are IQ scores, scores on standardized mathematics achievement tests, scores on mathematics aptitude tests, marks in previous grades or courses, and observations of teachers. It is common practice for the individual teacher to gather this information for himself by consulting the pupils' cumulative record cards and by giving such additional tests as he may desire.

There is apparently no standard, systematic, or continuous procedure for finding and listing pupils who have high achievement records in mathematics or marked potential ability in this area. Most

schools seem to seek to identify the superior pupil only when some special project is being considered such as the formation of a class for a possible enriched or accelerated program. Except where such special programs are set up, it is assumed that each teacher accepts the responsibility for identifying and guiding the superior pupils that are enrolled in his classes.

2. *What provisions are made for enriching the program of the superior pupil?*

Where superior pupils are in regular classes, teachers encourage or require them to undertake *special assignments*. These kinds of activities were reported in both junior and senior high schools as:

- doing library research work
- reading for interest or recreation
- writing book reports
- giving oral reports
- solving additional and more difficult problems
- investigating topics of special interest
- making models
- preparing exhibits for science fairs
- making scrapbooks
- watching and reporting on mathematics television programs
- organizing and conducting meetings of a mathematics club

Among additional activities reported for the superior pupil in the senior high school were:

- preparing lessons to teach to their classes
- conducting class discussions
- writing mathematics term papers
- meeting in groups to review for college entrance examinations
- competing in the national mathematics contest
- attending special Saturday morning classes at the University of Minnesota

In some of the junior high schools the *grouping* of pupils either by classes or within classes is being tried. While not all of the members of a "high" section are likely to meet the criteria for classification as "superior," the curriculum for these pupils is *enriched* by including topics which are outside the scope of the usual course and also not normally a part of any following course.

3. *What provisions are made for accelerating the program of the superior pupil?*

Two plans for acceleration are currently being tried:

> The able pupil may take in grade 8 the work in elementary algebra which is normally given in grade 9. He then continues through the standard sequence of plane geometry, advanced algebra, solid geometry, and trigonometry, and completes the regular high school program at the end of grade 11.
>
> After taking elementary algebra in grade 9 as usual, the superior pupil is given a course in combined plane and solid geometry in grade 10, followed by a course in combined advanced algebra and trigonometry in grade 11.

Under both plans it is proposed to give these accelerated students an opportunity to study college algebra and analytical geometry in grade 12. Advanced placement and college credit would be possible on the basis of examinations given either by the University of Minnesota, the College Entrance Examination Board, or the college to which the student applies for admission.

Most junior high schools report that able pupils are often accelerated on an individual basis. For example, a top student in grade 7 may be placed in algebra in grade 8; or a good student in grade 8 may study algebra under an individual instruction plan.

There is a trend, however, toward selecting pupils to form accelerated classes. During 1957–58, classes whose goal was to complete the standard arithmetic work of grades 7 and 8 in one year were reported at three junior high schools. Similar programs are being planned for 1958–59 at five schools. In each case the students who complete this one-year program success-

fully will study elementary algebra in grade 8. To continue the accelerated program for these pupils, some of the junior high schools will have a class in plane geometry in grade 9.

In the senior high schools, acceleration for able twelfth-grade pupils has always been available by teachers encouraging and assisting individuals to take examinations exempting them from courses in college algebra and college trigonometry. During the last two years selected high school seniors have been taking college freshman mathematics courses in a Saturday morning program sponsored by the Institute of Technology at the University of Minnesota. The development of accelerated sequences in mathematics in most of the high schools will lead shortly to the offering of this college program as a regular twelfth-year course.

An analysis of the provisions for superior pupils in mathematics as reported by the chairmen of the mathematics departments leads to the following observations and recommendations:

> It seems clear that the teachers in the junior and senior high schools are not only aware of the need for special provisions for the superior pupil but are also meeting this need in a variety of ways.
>
> As is probably highly desirable, each school is making those adjustments which it feels are most effective in terms of its size, the composition of its student body, and the availability of teacher personnel.
>
> Under whatever plans—individual or group—for providing for the superior pupil a school chooses to operate, it would seem that provisions for the superior pupil might be more effective if some systematic, continuous program for identifying and listing students with high potential in mathematics were worked out.
>
> Implicit in any plan for enriched programs—as contrasted with acceleration—is the use of concrete, specific "extra" materials with both individuals and classes. The preparation of such materials for inclusion at appropriate points in the curriculum is a project which needs to be worked on.
>
> The variety of proposals for providing for the superior pupils makes it imperative that teachers and administrators recognize the importance of careful planning and organization. For every program involving grouping or acceleration there should be a clear statement of the purpose of the program, a description of the materials and instructional methods to be used, and a plan for the evaluation of the program.

Available booklist for school libraries

The regulations recently adopted by the Office of Education on the administration of Title III of the National Defense Education Act of 1958 seem to spell out quite clearly that *these funds may be used to purchase books on mathematics for school libraries.* Apparently, many administrators are unaware that "the acquisition of supplementary reading materials to strengthen the mathematical portion of school libraries" is an eligible and important project under the Act. If mathematics teachers are to take advantage of this vital opportunity, they must act vigorously and swiftly at both the local and the state levels. Please do call this provision to the attention of *your* administrative officers. The high school book list prepared by the National High School and Junior College Mathematics Club is still available to persons sending a stamped self-addressed envelope (number 10 size) to High School Book List, c/o Richard V. Andree, The University of Oklahoma, Norman, Oklahoma.

• POINTS AND VIEWPOINTS

A column of unofficial comment

Multitrack programs

by H. Van Engen, University of Wisconsin, Madison, Wisconsin

The influx of "all the children of all the people" into our schools during the first quarter of this century placed a heavy strain on the academic sequence of courses in algebra and geometry. The mathematics teachers of that time concluded, rightfully, that "all of the children" should not be required to meet the standards of excellence needed for passing such courses as first-year algebra.

The solution arrived at is a matter of common knowledge. Schools established different programs in mathematics, and in other subjects as well, for pupils with different levels of academic abilities. In many instances, courses with different vocational objectives were developed. Thus we are now in an era of general mathematics, consumer mathematics, practical mathematics, shop mathematics, and so forth. The fracturing of mathematics programs in many ways proved to be an easy solution for the school systems of their day, but is it the program that is needed for the last quarter of the twentieth century? Maybe not.

Psychology taught us long ago that twelve-year-olds learn at varying rates. In fact, the child's ability to grasp abstractions is not a "one-variable" function of his chronological age. There are things other than chronological age which determine intellectual abilities, capacities, and interests. In spite of this, the schools are predominantly organized on a chronological basis—all twelve-year-olds are in the seventh grade (reviewing the fractions and decimals taught in the elementary school). Is this a really sensible way to organize our schools, or is it just a convenient way to organize the schools?

Most certainly, when his educational program is determined, the child's ability to learn should be given more weight than at present. It is far from reasonable to think that all seventh-graders should be studying the same mathematics; however, the problem cannot be met by establishing multitrack programs in mathematics—a one-track program is needed, which consists of good mathematics taught so as to take into consideration the rate at which children can learn mathematics. In summary form, here are just a few of the characteristics of the content and *modus operandi* of the one-track program:

1. A continuous, twelve-year program should be in operation. The program should be an "open-ended" program. Students with the prerequisite ability should be able to study calculus, some aspects of modern algebra, and statistics.
2. Each student would master this program to the extent of his ability and interest. The slow learner in a junior high school might require three years to master a program that the more rapid learner masters in two years or even one and one-half years.
3. Some pupils would come to the junior high schools ready to begin a serious

study of algebra and geometry. Others would enter without having been introduced to common fractions or decimal fractions.

4. Stocks, bonds, insurance, and shop mathematics for the slow learner would not be found in the mathematics curriculum. The guiding thought would be that if pupils understand mathematics, then the *application* of mathematics, at the level they are capable of applying their knowledge, will be no problem. Once the pupil understands mathematics, its application will not be troublesome, even though he will be required to know a few specialized words used in the applications.

Other differences and similarities could be cited. Basically the differences between the program for the slow learner and that for the rapid learner should be one of degree and not of kind. Today, with our multitrack programs, the emphasis is on differences in kind.

The alternative to the multitrack program will not be easy to establish in our schools. Too many fifth-grade teachers become emotionally disturbed if the fourth-grade teacher teaches "fifth-grade mathematics." After all, what is left for the fifth grade if the fourth-grade teacher does it all? Fifth-grade teachers must become accustomed to teaching more than "fifth-grade" mathematics.

Administrators must become accustomed to ninth-graders taking advanced courses in high school. Teachers must become accustomed to a more flexible program in mathematics and more flexible teaching procedures.

Does logic and common sense lead us to any other conclusion?

In the conditions of modern life, the rule is absolute: the race which does not value trained intelligence is doomed.—*Alfred North Whitehead.*

That delicate sensitiveness to the touch of the illogical, to the limits of knowledge, and to the Presence of the As-Yet-Unknown, which it was the object of great mathematicians to confer on automatic mechanism, is too often destroyed in the human brain by rough and ready processes, adopted sometimes for the purpose of fixing the opinions of young people, sometimes for that of enabling them to pass examinations successfully in subjects which they do not really understand.—*Taken from* The Preparation of the Child for Science, *by M. E. Boole.*

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Basic General Mathematics, Margaret Joseph and Mildred Keiffer; consultant and general editor: John R. Mayor (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1958). Cloth, iv+458 pp.

The following quotes from the preface quite accurately describe the general content, nature, and purpose of this textbook.

"This book provides a sound course in general mathematics for secondary school pupils who do not wish to study a traditional course in algebra or who wish to defer such a study until a more solid foundation in arithmetic and general mathematics has been established."

"The selection of content is based on . . . the experience of the authors in teaching general mathematics, supervising mathematics programs, and working with curriculum committees . . . and also upon recommendations of state curriculum committees."

As these quotations imply, the content of this book is much like that of many other general mathematics textbooks. Relative emphasis on various topics is roughly indicated by the number of pages devoted to each. (Topics listed are not chapter titles.)

Operations of arithmetic, including per cent	108
Measurement: area and volume formulas	46
Informal geometry	50
Algebra	34
Graphs	28
Consumer mathematics (budgets, banking, insurance, taxes, transportation, sports)	185

A definite effort has been made to make the book attractive in appearance. It is well bound and has a colorful cover. Throughout the book, green background rectangles are provided for chapter titles and paragraph headings. "Key words" are printed in green. More than forty photographs are used to provide interest and motivation and to illustrate uses of mathematics in everyday living. Extensive use of diagrams in various combinations of green and black increases the eye appeal and also portray the mathematical principles.

Throughout the book careful consideration has been given to motivation and teachability. Each chapter begins with a brief paragraph emphasizing the importance of the topic being presented. Brief explanations of processes and, in some cases, sample solutions are given. Interspersed throughout each chapter are "Class Exercises" to be done in class, "Exercises" for more class work or for home study, and "Think Before You Answer" oral exercises. At the close of each chapter are the following:

"Recreations," largely suitable for the better students

"Resource Materials," a list of magazine, book, and encyclopedia references

"Chapter Review"

"Cumulative Review."

The order of chapters seems to have been worked out to provide an interesting start and a change of pace. Consumer topics tend to alternate with the more purely mathematical topics and the more interesting with the less interesting. No answers to exercises are given, and no mention is made regarding availability of a teacher's answer book.

In regard to certain details some possible improvements might be mentioned. Discussion of large numbers, up to hundred millions, appears on page 20. Perhaps more experience should be provided with one-, two-, and three-digit numbers before generalizing the place-value principle to include large numbers. Similarly discussion of decimal fractions, which appears on page 25, might well be postponed. For the various computation algorithms only the "final" forms are shown; a more developmental procedure might be more meaningful. Only the "common denominator method" of division by a common fraction is presented. The "invert the divisor method" is probably the one with which most students will have had previous experience and it is just as easy to make meaningful. More precise definitions of "average" and "mean" would seem to be in order, even at this elementary level, particularly since "median" and "mode" are considered. In the chapter on algebra it might have been well to include some consideration of negative numbers, at least as an optional topic.

Taken as a whole, the book seems to be very well done, and adaptable to quite a range of ability levels. It merits careful consideration by any teacher or group looking for a textbook in general mathematics for secondary school pupils.—*Edwin Eagle, San Diego State College, San Diego, California.*

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

On the scoring of true-false tests

Robert S. Fouch

In the recent announcement of a national competitive mathematics examination, it is stated that "To discourage guessing, students are penalized for incorrect answers." This statement raises several interesting questions about the scoring and construction of true-false (and other multiple-choice) tests. One such question is whether the student is really "penalized" for guessing by computing his score as the number right minus the number wrong, rather than as the simple number right.

A complete analysis of this and related problems can become quite lengthy and laborious if all relevant factors are considered. Therefore, in this brief discussion a number of simplifying assumptions will be made. One of these, as an example, is that *guessing* is a completely random process, i.e., that answers are obtained by some method such as tossing a coin. Second, it will be assumed that the ordinary laws of probability apply whenever answers are obtained by guessing.

Another preliminary consideration is a somewhat unusual one, but seems important to the extent of being essential to any sound analysis of true-false tests (and also to most other types of evaluation). Ordinarily, we are content to say that a student knows or does not know a certain piece of information. However, a more detailed analysis of this is very much in order and can be achieved by means of an example of an item, "The sum of the interior angles of a triangle is 180°." There are clearly *three* possibilities concerning a student's knowledge on this topic: (1) he may know that this item is true; (2) he may be ignorant about the sum of these angles (and know that he is ignorant); (3) he may believe that the sum of these angles is some other number, say, 360°. For convenience, let us say that in case (1), the student has *positive information;* that in case (2), he has *complete ignorance;* that in case (3), he has *negative information.* It is the contention here that negative information is definitely worse than ignorance and that the scoring of any test should ideally make provision for this comparative evaluation of ignorance and negative information. It is further contended that the scoring "right−wrong" achieves such an evaluation.

To make this point, let us consider a hypothetical test and a sample of students for whom we have somehow obtained precise information about what they actually know concerning the items of this test. Suppose that there are 12 true-false items, that each item is a simple question of information, and that the items are of equal difficulty and importance.

The next step in this analysis is the difficult one of comparative evaluation of the learnings of various students. In particular, consider the problem of students D, E, and F. Clearly, D has more information than F, but how shall we compare D and F with E? To answer such questions, it is suggested that a value of $+1$ be assigned

Student	Number of items with: positive information	complete ignorance	negative information	Net information	Test results: Right	Wrong	R−W	Ranking: Theoretical	Right	R−W
A	12	—	—	+12	12	0	12	1	1	1
B	11	—	1	+10	11	1	10	2	2	2
C (guesses)	8	4	—	+ 8	10	2	8	3	3	3
D (doesn't guess)	8	4	—	+ 8	8	0	8	3	6	3
E	9	—	3	+ 6	9	3	6	5	4	5
F (guesses)	6	6	—	+ 6	9	3	6	5	4	5
G (guesses)	4	8	—	+ 4	8	4	4	7	6	7
H (guesses)	—	12	—	0	6	6	0	8	8	8
I (doesn't guess)	—	12	—	0	0	0	0	8	9	8
J	—	—	12	−12	0	12	−12	10	9	10

to each item of positive information, 0 to each item of complete ignorance, and −1 to each item of negative information. Next, it is suggested that we introduce the concept of *net information*, which we define as the algebraic sum of the values assigned above. As far as net information is concerned, we see that E and F obtain the same score. Finally, it is suggested that this net information score is a fair method of ranking students. Such a ranking of these 10 students is given in the ranking column under the heading "Theoretical."

Let us now see how each of these students will fare if he is ranked on the basis of number of test items right and of right − wrong. It should be a matter of no surprise that the right − wrong ranking is exactly the same as the theoretical ranking. Examination of the ranking based on the number right reveals several cases of what would almost certainly be called injustice. In particular, consider the rankings of students D and G, and also the rankings of I and J.

In conclusion, then, we would claim that the right − wrong scoring does not penalize guessing but, instead, serves to cancel out the effect of guessing and thus to treat fairly both the student who guesses and the one who chooses not to guess (witness students C and D and students H and I).

The reader may not find himself satisfied by this analysis. If so, it is likely to be due to the assumptions that were made in order to produce a simple theory. Further analysis is possible and would include such factors as: construction of the test in such a way that internal clues are minimized, logical analysis of the structure of the test items, more complicated probability considerations, and even the introduction of some of the ideas of game theory.

Heretofore Soviet schoolbooks on mathematics and science have been distinguished by the formality, directness, and economy of the text, emphasizing theoretical concepts and using only a limited number of examples, largely of abstract rather than applied nature.—*Alexander G. Korol,* Soviet Education for Science and Technology, p. 75.

• **TIPS FOR BEGINNERS**

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Review lessons in arithmetic

by Julia Adkins, Central Michigan College, Mount Pleasant, Michigan

Review lessons are essential in the teaching of mathematics, but too often they are merely repetition of previous work—the work being done in exactly the same way as originally presented. For a review to be successful it should approach a topic from a fresh point of view. The areas of history of mathematics, recreational mathematics, and applications of mathematics offer a wealth of material which may be used to provide this fresh point of view.

The philosophy expressed in the above paragraph may be applied at any level of teaching, but for the purpose of illustration this article will be limited to a discussion of a review of multiplication of whole numbers.

The introductory unit in most junior-high arithmetic books includes a review of the fundamental operations. If the review of the multiplication of whole numbers consists of problems in which multiplication is done in strange and unusual ways, followed by an analysis of these methods, the students will receive many benefits. Not only will they enjoy this type of work, but the meaning of multiplication will be clarified.

In the sections below, two of the more interesting and unusual historical methods of performing multiplication are discussed, then a few additional methods, with references, are listed.

Gelosia method

This method of multiplication had many names, but the one most commonly used was *gelosia.* The Italians used this name because the arrangement of the work resembled the grating (*gelosia*) placed at the windows of homes in Italian cities to prevent ladies or nuns from being easily seen (the word also meant "jealousy"). Other names occasionally used were: the lattice method, the grating method, the method of the net, and *shabacah* (an Arabic word meaning "network").

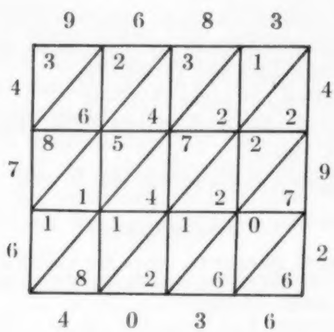
Many books of the fifteenth and sixteenth centuries included this method, but the writers were not in agreement as to the direction in which the diagonals of the cells should slant. The favorite method was that shown in Figure 1, on page 211.

Problem: 492×9683

Method:

1. Write the multiplicand across the top and the multiplier down the right side. (The answer is written across the bottom and up the left side.)
2. Write the digital products so that the unit's digit is in the lower half of each cell, and the ten's digit is in the upper half.
3. Add the digital products by beginning at the lower right-hand corner and proceeding to the left, adding along the diagonals.

a. The first diagonal contains one numeral (6).
b. The next diagonal to the left contains the numerals 7, 0, and 6. Their sum is 13. Write the 3 at the bottom of the diagonal, and "carry" the 1 to be added with the numerals along the third diagonal (these numerals are 2, 2, 2, 1, and 2).
c. Continue in this manner until the numerals along all of the diagonals have been added.



(492×9683 = 4,764,036)

Figure 1

Suggestions for use:

1. Have the students work a multiplication problem using the *gelosia* method, then check their answer by using the standard algorithm.
2. Have the students compare the two methods in order to determine *why* the *gelosia* method gives the correct answer. If the standard algorithm were arranged as shown in Figure 2, the comparison would be easier to make. With this arrangement the students will see that the numerals in the vertical columns of Figure 2 are the same as the numerals in the diagonal columns of Figure 1.

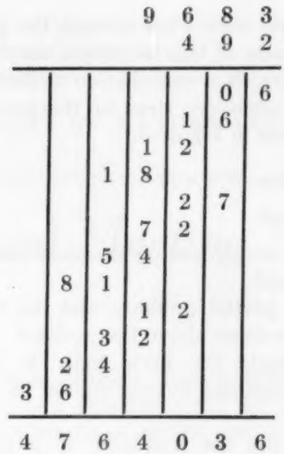


Figure 2

3. The teacher may use a problem done by the *gelosia* method for the purpose of analyzing a student's difficulties. This method shows more clearly if the student is having trouble with his multiplication combinations or with his addition combinations.

Scratch method

The scratch method was a European adaptation of a method used by the Hindus and by the Arabs (c. 1025). The strangeness of the method will be better understood when one recalls the types of writing materials available in early times. The Hindus wrote on boards strewed with sand, on white tablets strewed with red flour, and on blackboards (using a cane pen dipped in white paint). The boards were so small and their numerals were so large that they had to devise a technique by which only the absolute minimum of numerals was retained on the board. As soon as they ceased to need a numeral, they erased it.

After the Europeans began using paper they revised the early "erasure" method. Instead of erasing a numeral after it was

used, they drew a line through the numeral. Because of this technique they called it the scratch or cancellation method.

The successive steps of the procedure are shown in Figure 3.

Problem: 374×29

Method:

1. The multiplication proceeds from left to right.
2. The partial products and the answer are written above the problem.
3. Multiply the ten's digit (2) of the multiplicand by each digit of the multiplier.
4. Rewrite the multiplier in a position which is one place to the right of its former position.
5. Multiply the unit's digit (9) of the multiplicand by each digit of the multiplier.

		7	7
	6	$\cancel{6}4$	$\cancel{6}48$
29	29	29	29
374	$\cancel{3}74$	$\cancel{3}\cancel{7}4$	$\cancel{3}\cancel{7}\cancel{4}$
	a	*b*	*c*

0	08	084
71	$7\cancel{1}1$	$7\cancel{1}\cancel{1}$
$1\cancel{6}\cancel{4}8$	$1\cancel{6}\cancel{4}\cancel{8}$	$1\cancel{6}\cancel{4}\cancel{8}6$
29	29	29
$\cancel{3}7\cancel{4}4$	$\cancel{3}7\cancel{4}4$	$\cancel{3}7\cancel{4}\cancel{4}$
$\cancel{3}7$	$\cancel{3}7$	$\cancel{3}7$
d	*e*	*f*

(374×29=10846)

Figure 3

Suggestions for use:

Use the scratch method in a way similar to that suggested for the *gelosia* method. But with this method, help the students to see that each of the steps of Figure 3 consists of the following operations:

step *a*	$300\times20 = 6000$
step *b*	$6000+(70\times20) = 7400$
step *c*	$7400+(4\times20) = 7480$
step *d*	$7480+(300\times9) = 10180$
step *e*	$10180+(70\times9) = 10810$
step *f*	$10810+(4\times9) = 10846$

Other methods

There are many other historical and recreational methods, as well as applications of multiplication, that would provide interesting and useful ways of reviewing this operation. Some of these other techniques are listed below:

Method of duplation or doubling (3:557–9; 7:29–30)
Multiplicative magic squares (5:557–9)
Multiplication in number bases other than base ten (4:392–5)
Casting out nines (1:77–9)
Use of rules for short cuts (2:58, 119–20; 8:135–7)
Napier's Rods (6:339–40; 9:202)

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NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Thirty-seventh Annual Meeting

"The eyes of Texas" will indeed be focussed on the National Council of Teachers of Mathematics from April 1 through April 4, 1959 when the 37th Annual Meeting of the Council will be held in the city of Dallas located in the Lone Star State. While headquarters for the convention will be in the Baker Hotel, sectional meetings are scheduled in both the Baker Hotel and the Adolphus Hotel just across the street.

To provide a genuine introduction to the spirit of the southwest, a "barbecue dinner," with entertainment, has been planned for Wednesday evening, April 1, and a cordial invitation is extended to all who plan to attend the convention. Come early and enjoy this ranch-style meal, scheduled to be held in the Texas State Fair Park, the home of the world's largest state fair and of the Cotton Bowl.

One of the high lights of the program will be the presentation of the Twenty-fourth Yearbook, *The Growth of Mathematical Ideas*. This volume, prepared under the editorial direction of Phillip S. Jones, promises to be another of the many outstanding contributions of the Council to the professional literature related to mathematics education. Under the general heading, "The Growth of Mathematical Ideas," five sessions, beginning Thursday afternoon and continuing through Saturday afternoon, have been arranged to consider the themes and concepts included in the Yearbook. Those who believe that mathematics is more than a disconnected sequence of subjects, such as arithmetic, algebra, and geometry, will be pleased with this emphasis on the continuity of mathematical concepts. Early in his experiences the child is introduced to such ideas as number and operation, relation and function, measurement and approximation, probability and statistics, proof and symbolism; and to nourish the healthy growth of these ideas from the early years in the elementary school through the later years of the secondary school and beyond is the responsibility of the mathematics teacher. These and other concepts considered in the Yearbook provide the kind of continuity which enriches the study of mathematics and gives it meaning and significance. Each of these five sessions will be led by the authors of the sections under consideration and will include a panel discussion by qualified teachers from elementary, junior, and senior high schools.

At the first of the three general sessions E. G. Begle, of Yale University, will discuss the program of the School Mathematics Study Group, the work that has already been done, and plans for the future. Recognizing that this and other curricular studies have long-range implications for the education of mathematics teachers, Kenneth E. Brown, Mathematics Specialist of the U. S. Office of Education, has made a comprehensive study of the background and preparation of those who are teaching mathematics now. He will present his findings at the second general session under the title "The Quali-

fications and Teaching Load of Mathematics Teachers." At the third general session, Robert Fisher of Ohio State University will discuss "The Development of the Mathematics Teacher for Tomorrow." The Saturday luncheon speaker is John W. McFarland, Superintendent of Schools, Houston, Texas, and he has selected "A Time of Opportunity for Mathematics Teachers" as his topic. L. D. Haskew, Vice President of the University of Texas, has often been referred to as "the finest speaker that ever came out of Texas"—and if he can make that kind of a record *outside* the state it is not difficult to imagine what he will do when he speaks *inside* the state at the banquet on Friday evening. His theme, "The Higher Education," may not mean much to you now but it will have a much richer significance once you have listened to his interpretation of these three words.

Included in the program are approximately thirty-five sections, planned to meet the varying needs and interests of all who attend the convention. Problems in the teaching of mathematics from the early grades of the elementary school through undergraduate work on the college level are considered in special sections under the direction of able and competent leaders in the field of mathematics education. Illustrative of such problems are "Geometry in the Elementary School Curriculum," "Developing Algebraic Concepts in Grades 7 and 8," "Co-ordinate Geometry in the Plane Geometry Course—What and How?" and "Concept Learning in Differential Calculus." The interest of teachers in "modern" mathematics is reflected throughout the program, and sections are included to consider such topics as "Number Pairs in the Elementary School," "The Impact of Modern Mathematics on the Mathematics Curriculum of the Seventh and Eighth Grades," "Presentation of Specific Topics in Elementary Algebra Using Modern Language," and "Modern Introductory Mathematics for College Freshmen."

Curriculum trends on all levels will be discussed by qualified speakers; experimental studies now in progress will be reported and their implications considered. Other special sections deal with a variety of topics, such as "Research in Mathematics Education," "The Role of Evaluation," "Cooperation with Industry," "Teaching by Television," "Mathematical Contests," "Teacher Education," and our relations with the teachers of mathematics in other countries.

Nor has the program neglected the large and increasing interest in the important problem of providing for the academically talented student. Consistent with this interest, the Student Forum is concerned with "Calculus for the Tenth Grade" and, in addition, sectional meetings are devoted to the discussion of "Programs of Acceleration" and "A Summer Program for Talented High School Students," along with such specific topics as "Enriched Material for Superior Students in High School Algebra" and "Freewheeeling with the Gifted."

Three laboratory sessions are available, one for elementary school teachers, one for junior high, and one for senior high school teachers. In each of these laboratories selected teaching aids will be developed and each participant will actually construct teaching devices which will be helpful in his own classroom. School exhibits of student projects will be stimulating and suggestive, while commercial exhibits will include the latest textbooks along with a wide variety of teaching equipment. Provision will also be made for those who are interested in previewing many of the latest films designed to improve the teaching of mathematics.

Of special interest to those fortunate persons who preregister for them will be the two lecture-demonstrations of the Remington-Rand Scientific Computer at Southern Methodist University. This is one of the largest installations on any university campus, and this opportunity should prove of great value to those who

wish to know more about "Univac." Opportunities for visits to Dallas schools and tours to other places of interest are also being arranged.

To be present at this convention is to enjoy the finest professional experience available to mathematics teachers. More than one hundred persons have program responsibilities. All of them are highly successful teachers and many of them are distinguished leaders in the field of mathematics education. More general and sectional meetings than at any previous convention have been planned. Texas is a state that refuses to think small and there are rumors abroad that the mathematics teachers of the Lone Star State are expecting to break the attendance record established at Cleveland in 1958. Not only are "the eyes of Texas" watching but so are the eyes of all those who have a deep and genuine interest in improving the quality of teaching in the mathematics classrooms of America.

Mathematics and industry—a report

by Marie S. Wilcox, Chairman, Committee on Co-operation with Industry

It is well known that important industries have offered and are offering scholarships, supporting institutes, and financing contests for students and teachers of mathematics. Many of these activities are national in scope and have received national publicity. Hundreds of instances of co-operation between industries, teachers, and students of mathematics are occurring over the country at the local level. The Committee on Co-operation with Industry of the National Council feels that if publicity were given to a number of these projects, other local groups might see in them possibilities for co-operation in their own locality. This is the first of a series of articles which will be used for this purpose.

In the summers of 1957 and 1958, the Door Foundation, thirteen industries, and the Loomis School at Windsor, Connecticut, sponsored the Pre-College Science Center. It brought together for a seven-week period twenty-four boys from public, private, and parochial schools in Connecticut. The basic course taken by all of the boys was arranged to emphasize the importance of mathematics to the understanding of science. Permission was given to have groups of the boys work on their own projects in the shops and laboratories of local industries. Students were assigned to company personnel. These men, referred to as industrial advisors, suggested problems and supervised the work of the boys in the plants.

The International Business Machines in Philadelphia is sponsoring for the second year an eight-week course in "Computer Concepts" for teachers of Philadelphia. The class meets two hours each week, and teachers are given in-service credit by the local school system.

Co-operation between industry and education in southern California is well organized. A high light of this organization was the conference held in 1957 which created the Southern California Industry-Education Council to co-ordinate local programs. This council has an office and an executive secretary and publishes a news-letter about industrial-education activities.

The council reports that industries, technical societies, and other community groups are furnishing information for counsellors and students, initiating or supporting science careers days, and helping with science fairs and with science talent search and science and mathematics club

activities. They are inviting teachers and students to visit industrial plants, to attend technical meetings, and to read technical journals. Some industries provide summer industrial experience for students and teachers or assist financially with workshops, summer institutes, and other advanced study projects. Technically trained personnel speak to student, teacher, and community groups.

At present, attention in Los Angeles is focused on two major activities. One is a program of demonstration lectures for science and mathematics classes under which a team of qualified personnel from industry brings vital and authoritative science experiences to the students. By using plant equipment they frequently are able to give demonstrations which would not be possible with school laboratory equipment. In the spring of 1958, 20 teams visited 21 schools and had contact with 116 classes in science and mathematics.

The other important project in Los Angeles is the provision of summer work experience fellowships for teachers. The teacher is not offered permanent employment, but the positions are of professional level and closely related to the subject area of the teacher. A follow-up workshop is held by the school system to see how the experiences of the summer can best be incorporated into the classroom teaching.

Los Angeles also reports that IBM conducted a workshop on digital computors for teachers during the two weeks between the close of school and the opening of summer school in that area.

Have you read?

BENSON, ARNE. "New and Powerful Techniques in Finding Gear-Train Ratios," *Machine Design*, September 18, 1958, pp. 167–172.

Here is a short article that will be of great interest to your students who have mathematical ability and an interest in mechanics. Beyond this, however, it is a good illustration of the power of graphical representation in a form not commonly considered. For example, Mr. Benson shows the lattice for mediation of conjugate fractions and how this covers the plane with a series of triangles. The areas of these triangles in turn represent all possible fractional solutions. These in turn give the gear-train ratios desired to the required precision. The author gives several examples your students will enjoy working through.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BUTLER, JAMES C. "Operations Research," *The Rotarian*, October 1958, pp. 20–22, 56–57.

You no doubt have read about "Operations Research," but this article will be of interest to your students who may not have heard of it. For example, an electronic brain can read *Gone with the Wind* in three minutes and work 100 years of arithmetic in one afternoon. Operations Research was formed 15 years ago and helped fight two hot wars and several cold ones. The world of today cannot operate on a trial-and-error basis. Operations Research provides more precise methods through the study of men and machines at work. It employs Scientists, Mathematicians, Economists, Psychologists, and Political Scientists. It gathers data and translates it into mathematical terms. The computers then provide the results. Operations Research is a world-wide activity. It has been called "mathematical horse sense." I'm sure your students will enjoy this article—and even you might. Borrow the magazine from a local Rotarian.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

HAWKINS, DAVID. "Mathematical Sieves," *Scientific American*, December 1958, pp. 105–112.

We often think of abstract mathematics as a pure and exact science, and yet on close examination we find many uncertainties in it. Two thousand years ago Eratosthenes built the first mathematical sieve which is essentially a simple arithmetical process. This article presents a complete illustration and carefully explains the process. It probably will surprise you to note that the method of today is almost identical with that of 2000 years ago. The only difference is that we have machines to do some of the muscle work. I think your students will like to study the author's models of prime numbers and random numbers. From these they may branch out on their own. This material illustrates the fact that a problem of infinite complexity requires an infinite length of time to solve.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

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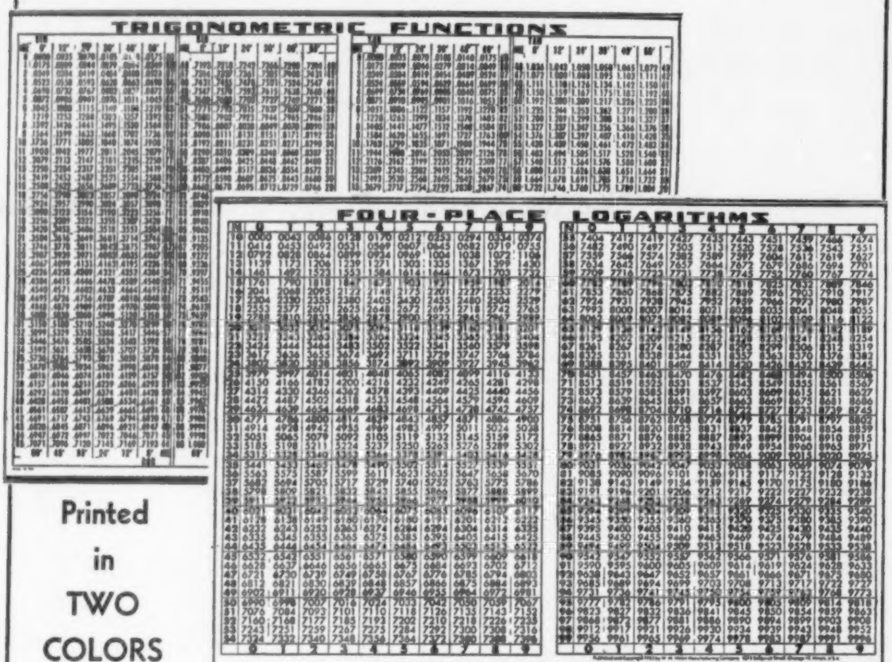

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